Errata list

Descriptive Set Theory and some applications to Functional Analysis

Thesis for the Degree of Licentiate of Philosophy

João Paulos, 2019

- Page 8, line 24, Th. 1.10 : $\mathbf{\Sigma}^{\mathbf{0}}_{\alpha} \subsetneq \mathbf{\Pi}^{\mathbf{0}}_{\alpha}$, **should** be $\mathbf{\Sigma}^{\mathbf{0}}_{\alpha} \nsubseteq \mathbf{\Pi}^{\mathbf{0}}_{\alpha}$.

- Page 26, line 24 and 25 , Th. 2.3 : *"In response, after P1 have played a_1, P2 will play some a_2 such that for any a_3 played by P2, (...)"*, **should** be *"In response, after P2 have played a_1, P1 will play some a_2 such that for any a_3 played by P2, (...)"*.

- Page 34, line 2, Remark 2.16 : $B = \varphi(A)$, **should** be $B = f^{-1}(\varphi(A))$.

- Page 46, line 1, Cor. 2.46 : *"(...) Wadge hierarchy is contained in at most (...)"*, **should** be *"(...) Wadge hierarchy contains at most (...)"*.

- Page 62, line 16, Prop. 3.13 : $S \sim c_n e^{inx}$, **should** be $S \sim \sum_n c_n e^{inx}$.

- Page 67, line 14, Cor. 3.28 : $\nu := \mu|_F$, **should** be $\nu := \lambda|_F$ and furthermore, $\nu(X) = \mu(X \cap F)$ **should** be $\nu(X) = \lambda(X \cap F)$.

- Page 88, line 10, Def. 4.28 : $\beta \leq \alpha$, **should** be $\beta < \alpha$.

- Page 90, line 19, Th. 4.38 : $\lambda.(\sup_i \kappa_i)$, **should** be $(\sup_i \kappa_i)^{\lambda}$.

- Page 91, line 32, Def. 4.43 : *(...) cofinal in β (...)*, **should** be *(...) cofinal in α (...)*.

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Descriptive Set Theory and some applications to Functional Analysis

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Classical Descriptive Set Theory and some applications to Functional Analysis

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Abstract

In this thesis, we study classical aspects of descriptive set theory with an emphasis on regularity properties and classification of complexity of certain sets. We introduce the Borel, projective, Wadge and Lipschitz hierarchies and study some of its properties. As an application, we present a collection of detailed examples concerning subsets of $C([0,1])$, sets of uniqueness and the point spectrum of a linear bounded operator acting on a separable Banach space.

Keywords : descriptive set theory, determinacy, differentiable functions, sets of uniqueness, point spectrum

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Preface

In this thesis, we study classical aspects of descriptive set theory with an emphasis on regularity properties and classification of complexity of certain sets. Roughly, the thesis is divided as follows :

• **Section 1.2** : Borel and projective hierarchies (and their basic properties), Suslin theorem (a set is Borel if and only if is analytic and coanalytic), existence of analytic sets which are not Borel.

• **Section 1.3** : Analytic sets have the perfect set property (PSP), the cardinality of any uncountable coanalytic is either $\aleph_1$ or $\mathfrak{c}$, analytic set have the Baire Property (BP) and are Lebesgue measurable (LM)

• **Section 2.1** : Games, the AD implies LM (in ZF)

• **Section 2.2** : A proof of Borel determinacy

• **Section 2.3** : Lipschitz and Wadge hierarchies (and their properties), Wadge's Lemma, characterization of the (semi) well-ordering of the Lipschitz and Wadge hierarchies in zero dimensional Polish spaces, Γ-hardness

• **Section 2.4** : Under $\boldsymbol{\Sigma}^1_1$-determinacy, all coanalytic sets have the PSP, under the PD all projective sets have the BP

• **Section 3.1** : Important $\boldsymbol{\Pi}^1_1$-complete sets : WF, $\{K \in \mathcal{K}([0,1]) : K \subseteq \mathbb{Q}\}$, Hurewicz's theorem

• **Section 3.2** : The set of functions in $C([0,1])$ which are piecewise differentiable and the set of functions in $C([0,1])$ which are differentiable on cocountable sets, are $\boldsymbol{\Pi}^1_1$-complete sets

• **Section 3.3** : Trigonometric series and descriptive set theory : the set of closed sets of uniqueness is $\boldsymbol{\Pi}^1_1$-complete in $\mathcal{K}(\mathbb{T})$

• **Section 3.4** : Any bounded analytic set in $\mathbb{C}$ is the point spectrum of a linear bounded operator acting on a separable Banach space, the point spectrum of any $T \in \mathcal{L}(X)$ for X reflexive and separable Banach space is a F_σ set, if X is reflexive and separable Banach, then the set $\{F \in \mathcal{F}(X) : F$ is uncountable and $F \neq \ker(T - \lambda 1)\}$ is analytic but not Borel in the Effros Borel space

• **Appendix** : Ordinal numbers and a few words on models and independence, generalities about Polish spaces and trees, properties of the spaces $\mathcal{K}(X)$ and $\mathcal{F}(X)$ and a generalization of Bari's Theorem (under Martin's axiom).

Contents

1 Borel and projective hierarchies

1.1 Introduction

The roots of descriptive set theory can be traced back to the work of french analysts (Borel, Baire, Lebesgue and others) around the turn of the 20th century. Around this time, concerns about the foundations of mathematics and the metamathematical aspects of logics, started to play a role within the psyche of the mathematical community. There was, perhaps for the first time, a serious debate about matters of definability, the axiomatic foundations and worries about constructivism in mathematics.
In this landscape, it was natural to expect some level of scepticism regarding definitions that we take for granted today (at least at the level of syntax). The concept of a *function*, an arbitrary correspondence between objects, was perhaps too vague and abstract for the taste of many prominent mathematicians. As a consequence, there was an effort to at least make sense of *natural* classes of such correspondences, which have a more algorithmic flavour. Lebesgue defined a certain class of correspondences, which would be what intuitively one may expect to be a function (at the very least, in the sense that this class contains virtually any function used in analysis at the time). In this context, he claimed that a projection onto the real line of a Borel set of the plane, remains a Borel set. Suslin noticed this to be false and it is fair to say that this started the study of classical descriptive set theory.

Suslin called the projections of Borel sets analytic and showed that there are analytic sets which are not Borel. The study of analytic sets and their properties was continued by the Russian and Polish school of mathematics and several *nice* regularity properties were established. Soon enough, the class of projective sets was introduced by Lusin and Sierpinski, as a *natural* extension of the class of Borel and analytic sets. However, these were much more complicated objects and the teenage years of descriptive set theory quickly revealed a rich and deep theory, with important (meta)mathematical connections.

Godel's consistency proof of the Continuum Hypothesis established the first boundaries for what one could say about projective sets within the ZFC framework. As Moschovakis writes in [3], '(...) the logicians entered the picture in their usual style, as spoilers. There was, however, another parallel development which brought them in more substantially and in a friendlier role'. This development was the birth of recursion theory and Kleene's definability theory for subsets of ω established a remarkable analogy between concepts of descriptive set theory and ideas of recursion theory. These are not only interesting mathematical statements but also, if one takes in consideration the Church-Turing thesis, statements with deep philosophical interest. Nowadays, the study of the so called *lightface hierarchies*, is systematized in effective descriptive set theory and this revealed (and continues) to be a fruitful approach even for matters related with classical descriptive set theory. However, as Moschovakis writes in [3] : 'Powerful as they are, the methods from logic and recursion theory cannot solve the "difficulties of the theory of projective sets," since they too are restricted by the limitations of Zermelo-Fraenkel set theory.' This scenario lead Solovay to show for the first time that strong set theoretic hypothesis imply significant results about projective sets. Namely, the role of large cardinals was studied in serious connection with descriptive set theory.

Once again, this seemed not to be enough and '(...) The next step was quite unexpected, even by those actively searching for strong hypotheses to settle the old open problems.' (c.f [3]) It is at this point that determinacy axioms and the study of infinite games proved to be an extremly important tool. Martin and Moschovakis, used determinacy axioms to settle persistent old questions about all levels of the projective hierarchy. The subject of descriptive set theory is thus, very rich both from the point of view of a mathematician or a (meta)mathematician.
In this section we aim to define the basic (and old) concepts that fueled the start of this field of research.

1.2 The Borel and the Projective hierarchies

1.2.1 Borel hierarchy

A Polish space is a topological space which is completely metrizable and second countable. Throughout this thesis, we shall use several standard results about Polish spaces which can be found in the appendix. Moreover, it is assumed that the reader is familiar with the basics of the theory of ordinal numbers. Again, a short introduction can be found in the appendix.

Notation 1.1. Let $\mathcal{F}$ be a family of sets. Then, the family of countable unions of elements of $\mathcal{F}$ is denoted by $\mathcal{F}_\sigma$ and the family of countable intersections of elements of $\mathcal{F}$ is denoted by $\mathcal{F}_\delta$.

Recall that given a set X, a collection of subsets $\Sigma \subseteq \mathcal{P}(X)$ which contains X and is closed under complements and countable unions is said to be a σ-algebra.

Notation 1.2. Given a topological space X, the smallest σ-algebra on X which contains all opens sets of X, i.e the Borel algebra of X, is denoted by $\mathcal{B}(X)$.

Henceforth, we assume that X is a metrizable space. We define the following classes of subsets of X by transfinite recursion (for $1 \leq \alpha < \omega_1$) :

$$\mathbf{\Sigma^0_1}(X) = \{\mathcal{U} \subseteq X \text{ such that } \mathcal{U} \text{ is open}\}$$

$$\mathbf{\Pi^0_1}(X) = \{F \subseteq X \text{ such that } F \text{ is closed}\}$$

For $1 < \alpha < \omega_1$, we define :

$$\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}(X) = (\bigcup_{\beta<\alpha} \mathbf{\Pi}^0_{\boldsymbol{\beta}}(X))_\sigma$$

$$\mathbf{\Pi}^0_{\boldsymbol{\alpha}}(X) = (\bigcup_{\beta<\alpha} \mathbf{\Sigma}^0_{\boldsymbol{\beta}}(X))_\delta$$

The classes $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}(X)$ are called additive classes and the classes $\mathbf{\Pi}^0_{\boldsymbol{\alpha}}(X)$ are called multiplicative classes. This gives a stratification of the Borel sets of X in a hierarchy of at most ω_1 levels (see Theorem 1.4), the **Borel hierarchy**.

Note that we assumed X to be a metrizable space and thus, every open set is a F_σ-set. This is convenient since it follows immediately that $\mathbf{\Sigma}^0_{\mathbf{1}}(X) \subseteq \mathbf{\Sigma}^0_{\mathbf{2}}(X)$.

We will further omit any reference to X when referring to additive or multiplicative classes unless it is strictly necessary. The following is a collection of standard facts about additive and multiplicative classes :

Proposition 1.3. Let X be a metrizable space. Then :

(i) $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$ and $\mathbf{\Pi}^0_{\boldsymbol{\alpha}}$ are closed under finite unions and finite intersections. Moreover, $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$ is closed under countable unions and $\mathbf{\Pi}^0_{\boldsymbol{\alpha}}$ is closed under countable intersections.

(ii) For all $\omega_1 > \alpha \geq 1$, $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}} = \neg\mathbf{\Pi}^0_{\boldsymbol{\alpha}}$.

(iii) If $1 \leq \alpha < \beta < \omega_1$, then $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}} \subseteq \mathbf{\Pi}^0_{\boldsymbol{\beta}}$ and $\mathbf{\Pi}^0_{\boldsymbol{\alpha}} \subseteq \mathbf{\Sigma}^0_{\boldsymbol{\beta}}$.

(iv) If α is a limit ordinal with $\alpha = \lim_n \alpha_n$, then $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}} = (\bigcup_n \mathbf{\Pi}^0_{\boldsymbol{\alpha_n}})_\sigma$ and $\mathbf{\Pi}^0_{\boldsymbol{\alpha}} = (\bigcup_n \mathbf{\Sigma}^0_{\boldsymbol{\alpha_n}})_\delta$

(v) For all $\omega_1 > \alpha \geq 1$, $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$ and $\mathbf{\Pi}^0_{\boldsymbol{\alpha}}$ are closed under continuous preimages.

(vi) Let $\omega_1 > \alpha \geq 1$ and suppose $Y \subseteq X$, then $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}(Y) = \{A \cap Y, A \in \mathbf{\Sigma}^0_{\boldsymbol{\alpha}}(X)\}$ and $\mathbf{\Pi}^0_{\boldsymbol{\alpha}}(Y) = \{A \cap Y, A \in \mathbf{\Pi}^0_{\boldsymbol{\alpha}}(X)\}$.

Proof. The reader can find a proof in [2] (Proposition 3.6.1, p. 116). ■

Theorem 1.4. *For any metric space,* $\mathcal{B}(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_{\boldsymbol{\alpha}} = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_{\boldsymbol{\alpha}}$

Proof. It follows by definition that for any $\alpha < \omega_1$, $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}} \subseteq \mathcal{B}(X)$. It is enough to prove that $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$ is a σ-algebra. Given $B \in \mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$, then $X \setminus B \in \mathbf{\Pi}^0_{\boldsymbol{\alpha}}$ and thus, $X \setminus B \in \mathbf{\Sigma}^0_{\boldsymbol{\alpha+1}}$, hence $\mathcal{A}$ is closed under complements. Moreover, suppose that $\{B_n\} \subseteq \mathcal{A}$, with $B_n \in \mathbf{\Sigma}^0_{\boldsymbol{\alpha_n}}$. Taking $\alpha = \sup_n \alpha_n < \omega_1$ it follows by Proposition 1.3 (iv) that $\bigcup_n B_n \in \mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$. Hence, $\mathcal{A}$ is closed under countable unions. The case with multiplicative classes is entirely analogous. ■

With the stratification of the Borel sets given by Theorem 1.4, we can prove that if X is an infinite metric space which is separable then there are constraints on the cardinality of its Borel algebra. In order to prove this, we need the following result :

Proposition 1.5. Let X be a separable and infinite metric space. Then, there are $\mathfrak{c}$ many open sets in X.

Proof. Since X is a separable space, then it has a countable basis and thus, X has at most $\mathfrak{c}$ distinct open sets. On the other hand, we prove that X has at least $\mathfrak{c}$ distinct open sets and thus, $|\mathbf{\Sigma^0_1}(X)| = \mathfrak{c}$. This can be seen as follows: if X has infinitely many isolated points it is clear that there are at least $\mathfrak{c}$ distinct open sets. If X only has finitely many isolated points, since X is infinite then there are infinitely many non isolated points and one can, without loss of generality, assume that in this case X has no isolated points. We will define an open set $\mathcal{U}_s$ for each $s \in 2^{<\omega}$ such that if $s \perp t$, then $\mathcal{U}_s \cap \mathcal{U}_t = \emptyset$ (this is an example of a scheme. Suslin (and Lusin) schemes are very useful techniques in classical Descriptive Set Theory and they will be introduced properly in section 1.3). We start with $\mathcal{U}_\emptyset = X$ and since X has at least two points, we can choose two disjoint non empty open sets $\mathcal{U}_0$ and $\mathcal{U}_1$.
We define the remaining open sets by induction : suppose that we have defined for some $s \in 2^{<\omega}$ a non empty open set $\mathcal{U}_s$ with the property that whenever $s \perp t$ (and $\mathcal{U}_t$ is defined), then $\mathcal{U}_s \cap \mathcal{U}_t = \emptyset$. Since X is assumed to have no isolated points, there are distinct points $x_0, x_1 \in \mathcal{U}_s$ and one can choose disjoint open sets such that $x_0 \in \mathcal{U}_{s^\frown 0}$ and $x_1 \in \mathcal{U}_{s^\frown 1}$.
Finally, this will induce an injective map $2^\omega \hookrightarrow \mathbf{\Sigma^0_1}$: for $\alpha \in 2^\omega$, define :

$$\mathcal{V}^\alpha_0 = \mathcal{U}_{\alpha(0)} \text{ and for each } n > 1,\ \mathcal{V}^\alpha_n = \mathcal{U}_{(|1-\alpha(0)|,\ldots,|1-\alpha(n-1)|,\alpha(n))}$$

We let $\iota : 2^\omega \to \mathbf{\Sigma^0_1}$ such that :

$$\alpha \mapsto \bigcup_{n\geq 0} \mathcal{V}^\alpha_n$$

Since ι is injective, this proves that $|\mathbf{\Sigma^0_1}| \geq \mathfrak{c}$. ∎

Corollary 1.6. If X is an infinite metric separable space, then $|\mathcal{B}(X)| = \mathfrak{c}$

Proof. By Proposition 1.5, $|\mathbf{\Sigma^0_1}| = |\mathbf{\Pi^0_1}| = \mathfrak{c}$ and thus, for each $\alpha < \omega_1$, $|\mathbf{\Sigma^0_\alpha}| = |\mathbf{\Pi^0_\alpha}| = \mathfrak{c}$. The result follows immediately from Theorem 1.4, since $\omega_1 \leq 2^\omega$ and $\mathfrak{c} \times \mathfrak{c} = \mathfrak{c}$. ∎

Remark 1.7. Corollary 1.6 provides a simple proof of the fact that there are Borel sets of the real line $\mathbb{R}$ which are not Lebesgue measurable. Indeed, since the Lebesgue measure μ is complete and $\mu(2^\omega) = 0$ - where $2^\omega \subseteq \mathbb{R}$ is the Cantor set - then, it follows that there are at least $|2^{2^\omega}|$ Lebesgue measurable subsets of $\mathbb{R}$, while there are only $|2^\omega|$ many Borel sets.

1.2.2 Universal sets

According to Theorem 1.4, if X is a metric space then we can stratify $\mathcal{B}(X)$ in a hierarchy with at most ω_1 levels. We prove that, if X is an uncountable Polish space, then every such level is needed (Theorem 1.10). In order to prove this, we introduce the concept of *universal set* for a pointclass. In what follows, $\alpha < \omega_1$ and Γ_α is either $\mathbf{\Sigma^0_\alpha}$ or $\mathbf{\Pi^0_\alpha}$.

Given non-empty sets X and Y and $U \subseteq X \times Y$, we define :

$$\text{For } x \in X,\, U_x = \{y \in Y : (x,y) \in U\}$$

$$\text{For } y \in Y,\, U^y = \{x \in X : (x,y) \in U\}$$

Definition 1.8. Let X and Y be metrizable spaces. We say that $U \in \Gamma_\alpha(X \times Y)$ is X-universal for $\Gamma_\alpha(Y)$ if for every $A \in \Gamma_\alpha(Y)$, there is some $x \in X$ such that $A = U_x$.

In a sense, universal sets *parametrize* certain pointclasses.

Notation 1.9. An ubiquitous Polish space throughout this thesis is the product space ω^ω, with ω endowed with the discrete topology. The set of finite sequences on ω, will be denoted by $\omega^{<\omega}$. Given any $s \in \omega^{<\omega}$, we denote the associated basic open set by $\Sigma(s) = \{x \in \omega^\omega : x(n) = s(n) \text{ for } n \leq |s|\}$.

Theorem 1.10. *Let X be an uncountable Polish space. Then, $\mathbf{\Sigma^0_\alpha} \subsetneq \mathbf{\Sigma^0_{\alpha+1}}$ for any $\alpha < \omega_1$.*

Proof. The result follows from the following claim :

Claim 1 : Let X be an uncountable Polish space. Then, there is a $U \in \mathbf{\Sigma^0_\alpha}(X \times X)$ which is X-universal for $\mathbf{\Sigma^0_\alpha}(X)$.

Assuming Claim 1, let X be any uncountable Polish space and suppose that $U \subseteq X \times X$ is X-universal for $\mathbf{\Sigma^0_\alpha}(X)$. Let $A = \{x \in X : (x,x) \in U\}$. Since $\mathbf{\Sigma^0_\alpha}$ is closed under continuous preimages we have that $A \in \mathbf{\Sigma^0_\alpha}$. If $A \in \mathbf{\Pi^0_\alpha}$, then let $x \in X$ such that $X \setminus A = U_x$. But this is impossible, since :

$$x \in X \setminus A = U_x \text{ iff } (x,x) \in U \text{ iff } x \in A$$

Hence, $\mathbf{\Sigma}^\mathbf{0}_\alpha \subsetneq \mathbf{\Pi}^\mathbf{0}_\alpha$ and it follows, by definition of additive classes, that $\mathbf{\Sigma}^\mathbf{0}_\alpha \subsetneq \mathbf{\Sigma}^\mathbf{0}_{\alpha+\mathbf{1}}$. Thus, it remains to prove Claim 1 and in order to do so, we need the following result :

Claim 2 : Let Y be any second countable metrizable space. Then, there is some $U \in \mathbf{\Sigma^0_\alpha}(\omega^\omega \times Y)$ which is ω^ω-universal for $\mathbf{\Sigma^0_\alpha}(Y)$ (and similarly for $\mathbf{\Pi}^\mathbf{0}_\alpha(Y)$).

Proof of Claim 2: We prove this by induction on α.

We start with $\alpha = 1$. Let $\{\mathcal{V}_n\}$ be a countable basis for Y and, without loss of generality, assume that $\mathcal{V}_0 = \emptyset$. Define $U \subseteq \omega^\omega \times Y$ by :

$$(x, y) \in U \text{ if and only if } y \in \bigcup_n \mathcal{V}_{x(n)}$$

We check that $U \in \mathbf{\Sigma^0_1}(\omega^\omega \times Y)$ is ω^ω-universal for $\mathbf{\Sigma^0_1}(Y)$:

Indeed, U is open : let $(x_0, y_0) \in U$ so that by definition there is some n such that $y_0 \in \mathcal{V}_{x_0(n)}$. Hence, $(x_0, y_0) \in \{x \in \omega^\omega : x(n) = x_0(n)\} \times \mathcal{V}_{x_0(n)} \subseteq U$. Furthermore, let $W \subseteq Y$ be any open subset. If $W = \emptyset$, then $W = U_x$ for $x = (0, 0, ...)$. Otherwise, $W = \bigcup_j \mathcal{V}_{i_j}$ and we let $x \in \omega^\omega$ such that $x(j) = i_j$. It follows that $W = U_x$. We conclude that U is ω^ω-universal for $\mathbf{\Sigma^0_1}(Y)$ and consequently, $\omega^\omega \times Y \setminus U$ is ω^ω-universal for $\mathbf{\Pi^0_1}(Y)$.

Now, let $\alpha = \lim_n \alpha_n$ be a limit ordinal. By induction hypothesis, there are ω^ω-universal sets for $\mathbf{\Pi^0_{\alpha_n}}(Y)$, say U_{α_n}. We fix a bijection $\langle n, m\rangle$ between ω^2 and ω and given $x \in \omega^\omega$, we define an element $(x)_n \in \omega^\omega$ such that $(x)_n(m) = x(\langle n, m\rangle)$. Define $U \subseteq \omega^\omega \times Y$ such that :

$$(x, y) \in U \text{ if and only if } \exists n \text{ such that } ((x)_n, y) \in U_{\alpha_n}$$

We check that U is ω^ω-universal for $\mathbf{\Sigma^0_\alpha}(Y)$:

Consider the continuous map $f_n : (x, y) \mapsto ((x)_n, y)$. Since $U = \bigcup_n f_n^{-1}(U_{\alpha_n})$, it follows that $U \in \mathbf{\Sigma^0_\alpha}(\omega^\omega \times Y)$. Furthermore, let $A \in \mathbf{\Sigma^0_\alpha}(Y)$ such that $A = \bigcup_n A_n$, with $A_n \in \mathbf{\Pi^0_{\alpha_n}}(Y)$ and $A_n = (U_{\alpha_n})_{x_n}$. We note that there is some $z \in \omega^\omega$ such that for every n, one has that $(z)_n = x_n$, since $\langle ., .\rangle$ is a bijection. It follows, by definition, that $A = U_z$. Consequently, $\omega^\omega \times Y \setminus U$ is ω^ω-universal for $\mathbf{\Pi^0_\alpha}(Y)$.

It remains the case when $\alpha = \beta + 1$. Let V be ω^ω-universal for $\mathbf{\Pi^0_{\beta}}(Y)$, which exists by induction hypothesis. Define $U \subseteq \omega^\omega \times Y$ such that :

$$(x, y) \in U \text{ if and only if } \exists n \text{ such that } ((x)_n, y) \in V$$

Similarly with the previous case, we get that U is ω^ω-universal for $\mathbf{\Sigma^0_{\alpha}}(Y)$ and, consequently, that $\omega^\omega \times Y \setminus U$ is ω^ω-universal for $\mathbf{\Pi^0_{\alpha}}(Y)$.

We finally prove Claim 1 :

Proof of Claim 1: Since X is uncountable, it follows from Theorem 4.65 that there is some $Y \subseteq X$ which is homeomorphic to ω^ω. By Claim 2, there is some $U \subseteq Y \times X$ which is Y-universal for $\mathbf{\Sigma^0_{\alpha}}(X)$. By Proposition 1.3 (vi), there exists some $V \in \mathbf{\Sigma^0_{\alpha}}(X \times X)$ such that $U = V \cap (Y \times X)$. Clearly, V is X-universal for $\mathbf{\Sigma^0_{\alpha}}(X)$. Its complement is X-universal for $\mathbf{\Pi^0_{\alpha}}(X)$. ■

Remark 1.11. We note that if X is an uncountable Polish space, then a similar argument given in the proof of Theorem 1.10 also shows that $\mathbf{\Pi^0_\alpha}(X) \subsetneq \mathbf{\Pi^0_{\alpha+1}}(X)$.

1.2.3 Projective hierarchy

Definition 1.12. Let X be a Polish space and $A \subseteq X$. We say that A is **analytic** if there is some Borel set $B \subseteq X \times X$ such that $A = \pi_X(B)$. We denote the class of analytic sets of X by $\mathbf{\Sigma}^1_1(X)$. The complement of an analytic set is said to be a **coanalytic** set and we denote the class of coanalytic sets of X by $\mathbf{\Pi}^1_1(X)$.

The following is an useful and well-known characterization of analytic sets :

Theorem 1.13. *Let X be a Polish space and $A \subseteq X$. The following are equivalent :*

(i) A is an analytic set.

(ii) There is a Polish space Y and a Borel subset $B \subseteq X \times Y$ whose projection is A.

(iii) There is a continuous map $f : \omega^\omega \to X$, whose range is A.

(iv) There is a closed subset $C \subset X \times \omega^\omega$ whose projection is A.

(v) For every uncountable Polish space Y, there is a G_δ set $G \subseteq X \times Y$ whose projection is A.

Proof. The reader can find a proof in [2] (Proposition 4.1.1, p. 128). ∎

Remark 1.14. Another equivalent definition for an analytic set of a Polish space X, which is frequently found in the literature (and used in this thesis) is the following :

$$A \subseteq X \text{ is analytic iff } A \text{ is the continuous image of a Polish space}$$

Indeed, if A is analytic then (by Theorem 1.13) it is a continuous image of ω^ω, which is a Polish space. On the other hand, suppose that there is a Polish space Y and a continuous map $f : Y \to X$ such that $f(Y) = A$. By Theorem 4.70, it follows that A is a continuous image of ω^ω and thus, by Theorem 1.13, A is analytic.

The following result, is a collection of standard, yet useful, properties about analytic sets :

Proposition 1.15. Let X be a Polish space. Then :

(i) The class $\mathbf{\Sigma}^1_1(X)$ is closed under countable unions and countable intersections.

(ii) Let Y be a Polish space and $f : X \to Y$ be a Borel map. If $A \in \mathbf{\Sigma}^1_1(X)$ and $B \in \mathbf{\Sigma}^1_1(Y)$, then $f(A) \in \mathbf{\Sigma}^1_1(Y)$ and $f^{-1}(B) \in \mathbf{\Sigma}^1_1(X)$.

Proof. The reader can find a proof in [1] (Proposition 14.4, p. 86). ∎

The classes $\mathbf{\Sigma^1_1}(X)$ and $\mathbf{\Pi^1_1}(Y)$ define the *first* level of the so called **projective hierarchy**. For $n \geq 1$ we define the projective sets in the following manner :

$$A \in \mathbf{\Sigma^1_{n+1}}(X) \text{ if there is some } B \in \mathbf{\Pi^1_n}(X \times X) \text{ such that } A = \pi_X(B).$$

$$\mathbf{\Pi^1_{n+1}}(X) = \neg\mathbf{\Sigma^1_{n+1}}(X)$$

If we denote by $\mathbf{P}$ the class of projective sets, one can verify that :

$$\mathbf{P} = \bigcup_n \mathbf{\Sigma^1_n}(X) = \bigcup_n \mathbf{\Pi^1_n}(X)$$

The role of projections is apparent in the definition of the projective sets. It is then worth to set the following notation :

Notation 1.16. Let X, Y be any sets and let $B \subseteq X \times Y$. We then define $\exists^Y B = \{x \in X : (x, y) \in B, \text{ for some } y \in Y\}$, the projection of B. We also define $\forall^Y B = \{x \in X : (x, y) \in B, \text{ for all } y \in Y\}$, the coprojection of B.

The following collection of facts can be seen as an analogous of Proposition 1.3, but now for the *whole* projective hierarchy (as we did with the Borel hierarchy, when referring to projective classes we often omit the underlying space in our notation) :

Proposition 1.17. Let X be a Polish space. Then :

(i) $\mathbf{\Sigma^1_n}$ and $\mathbf{\Pi^1_n}$ are closed under countable unions, countable intersections and Borel preimages.

(ii) $\mathbf{\Sigma^1_n}$ is closed under $\exists^Y$ and $\mathbf{\Pi^1_n}$ is closed under $\forall^Y$, for any Polish space Y.

(iii) For all $n \geq 1$, $\mathbf{\Sigma^1_n} \subseteq \mathbf{\Sigma^1_{n+1}}$, $\mathbf{\Sigma^1_n} \subseteq \mathbf{\Pi^1_{n+1}}$, $\mathbf{\Pi^1_n} \subseteq \mathbf{\Pi^1_{n+1}}$ and $\mathbf{\Pi^1_n} \subseteq \mathbf{\Sigma^1_{n+1}}$.

Proof. The reader can find a proof in [2] (Proposition 4.1.7 and Proposition 4.1.9, p. 131) . ■

1.2.4 Lusin separation theorem and Suslin theorem

Definition 1.18. Let X be a Polish space. For any $1 \leq \alpha < \omega_1$ and $n \geq 1$, we define the so called ambiguous classes as follows :

$$\mathbf{\Delta^0_\alpha} = \mathbf{\Sigma^0_\alpha} \cap \mathbf{\Pi^0_\alpha}$$

$$\mathbf{\Delta^1_n} = \mathbf{\Sigma^1_n} \cap \mathbf{\Pi^1_n}$$

Remark 1.19. Note that it follows from definition that $\mathbf{\Delta^1_1} \subseteq \mathbf{\Sigma^1_1}$.

In this section, we prove two fundamental results that relate ambiguous classes with the first level of projective sets on a Polish space X : the Lusin separation theorem (Theorem 1.21) and the Suslin theorem (Corollary 1.24).

It follows by the Lusin separation thorem that $\mathcal{B}(X) = \mathbf{\Delta^1_1}$ and thus, the class of Borel sets coincides with the class of analytic sets with analytic complement.

On the other hand, Lusin theorem establishes that the projective hierarchy extends the Borel hierarchy for uncountable Polish spaces. More concretely, we prove that if X is an uncountable Polish space, then $\mathbf{\Delta^1_1}(X) \subsetneq \mathbf{\Sigma^1_1}(X)$ (and so, in particular, not every projection of a Borel subset of the plane is a Borel set of the real line).

Definition 1.20. Let X be a Polish space and consider $A, B \subseteq X$, two disjoint subsets. We say that A and B are separated if there is some $C \in \mathcal{B}(X)$ such that $A \subseteq C$ and $B \cap C = \emptyset$.

Theorem 1.21. *Let X be a Polish space and $A, B \in \mathbf{\Sigma^1_1}$ which are disjoint. Then, A and B can be separated.*

Proof. We begin with a simple combinatorial claim :

Claim: Suppose $E = \bigcup_n E_n$ and $F = \bigcup_m F_m$ can't be separated. Then, there is some n and some m such that E_n and F_m can't be separated.

Proof of Claim: If for every n and m there is some $C_{nm} \in \mathcal{B}(X)$ such that $E_n \subseteq C_{nm}$ and $F_m \cap C_{nm} = \emptyset$, then take $C = \bigcup_n \bigcap_m C_{nm} \in \mathcal{B}(X)$, which separates E and F.

Now let A and B be two disjoint analytic subsets of X. Let $f : \omega^\omega \to A$ and $g : \omega^\omega \to B$ be continuous surjections. Suppose, by contradiction, that there is no Borel set C which separates A and B. We will construct inductively $\alpha, \beta \in \omega^\omega$ such that for every n, $f(\Sigma(\alpha|_n))$ can't be separated from $g(\Sigma(\beta|_n))$. This leads to a contradiction : Indeed, note that since $A \cap B = \emptyset$ we have that $f(\alpha) \neq g(\beta)$. Using that X is Hausdorff and that f and g are continuous, there are disjoint open sets $\mathcal{U} \in \mathcal{N}_{f(\alpha)}$ and $\mathcal{V} \in \mathcal{N}_{g(\beta)}$ and some N such that $f(\Sigma(\alpha|_N)) \subseteq \mathcal{U}$ and $g(\Sigma(\beta|_N)) \subseteq \mathcal{V}$. In particular, $f(\Sigma(\alpha|_N))$ and $g(\Sigma(\beta|_N))$ are separated.

In order to construct such $\alpha, \beta \in \omega^\omega$, note that $A = \bigcup_n f(\Sigma(n))$ and that $B = \bigcup_m g(\Sigma(m))$. Thus, it follows by our assumption and our Claim that there are $\alpha(0)$ and $\beta(0)$ such that $f(\Sigma(\alpha(0)))$ and $g(\Sigma(\beta(0)))$ can't be separated. Now, assume that we have already chose $\alpha(0), ..., \alpha(k)$ and $\beta(0), ..., \beta(k)$ satisfying the above conditions. Note that :

$$f(\Sigma(\alpha(0), ..., \alpha(k))) = \bigcup_n f(\Sigma((\alpha(0), ..., \alpha(k), n)))$$

$$g(\Sigma(\beta(0), ..., \beta(k)))) = \bigcup_m g(\Sigma((\beta(0), ..., \beta(k), m)))$$

To get $\alpha(k+1)$ and $\beta(k+1)$ we simply apply our Claim and our assumption once again. Hence, we have defined inductively $\alpha, \beta \in \omega^\omega$ with the desired property. $\blacksquare$

Corollary 1.22. Let X be a Polish space. Then, $\mathcal{B}(X) = \mathbf{\Delta^1_1}$.

Proof. On one hand, let $A \in \mathcal{B}(X)$. Since $B = A \times A \in \mathcal{B}(X \times X)$ it is clear that $\mathcal{B}(X) \subseteq \mathbf{\Sigma^1_1}$. Since $\mathcal{B}(X)$ is closed under complements, it follows that $\mathcal{B}(X) \subseteq \mathbf{\Pi^1_1}$. Therefore, $\mathcal{B}(X) \subseteq \mathbf{\Delta^1_1}$. On the other hand, let $A \in \mathbf{\Delta^1_1}$. Thus, $X \setminus A \in \mathbf{\Sigma^1_1}$ and it follows by Theorem 1.21 (using $B = X \setminus A$) that $A \in \mathcal{B}(X)$. $\blacksquare$

We finish this section with a proof that there is an analytic set which is not Borel, relying on diagonalization arguments and universal sets as we did previously for the Borel hierarchy. Given Polish spaces X and Y and $U \in \mathbf{\Sigma^1_1}(X \times Y)$, U is said to be X-universal for $\mathbf{\Sigma^1_1}(Y)$ if, given any $A \in \mathbf{\Sigma^1_1}(Y)$ there is some $x \in X$ such that $U_x = A$ and thus, we generalize the previous definition of universal sets.

Theorem 1.23. *Let X be an uncountable Polish space. Then, there is some $U \in \mathbf{\Sigma^1_1}(X \times X)$ which is universal for $\mathbf{\Sigma^1_1}(X)$.*

Proof. It is enough to prove that there is some $U \in \mathbf{\Sigma^1_1}(\omega^\omega \times X)$ which is ω^ω-universal for $\mathbf{\Sigma^1_1}(X)$. Indeed, if such set exists then we just appeal to Theorem 4.65 since X is uncountable and thus, contains a copy of ω^ω.
Note that we proved that there is some $C \subseteq \omega^\omega \times (X \times \omega^\omega)$ which is ω^ω-universal for $\mathbf{\Pi^0_1}(X \times \omega^\omega)$ while proving Theorem 1.10 (*Claim 2*). Let :

$$U = \{(\alpha, x) \in \omega^\omega \times X : (\alpha, x, \beta) \in C \text{ some } \beta\}$$

Since $U = \exists^{\omega^\omega} C$, it follows from Proposition 1.17 (ii) that $U \in \mathbf{\Sigma^1_1}(\omega^\omega \times X)$. Furthermore, let $A \in \mathbf{\Sigma^1_1}(X)$. By Theorem 1.13 (iii), there is some $F \subseteq X \times \omega^\omega$ such that $\pi_X(F) = A$. Let $\alpha \in \omega^\omega$ such that $C_\alpha = F$. Then, $A = U_\alpha$:

$$x \in U_\alpha \Leftrightarrow \exists \beta \text{ such that } (\alpha, x, \beta) \in C \Leftrightarrow (x, \beta) \in C_\alpha \Leftrightarrow x \in A$$

$\blacksquare$

Corollary 1.24. Let X be an uncountable Polish space. Then, $\mathbf{\Delta^1_1} \subsetneq \mathbf{\Sigma^1_1}$.

Proof. Let $U \in \mathbf{\Sigma^1_1}(X \times X)$, universal for $\mathbf{\Sigma^1_1}(X)$. By Proposition 1.17 we get that $A = \{x \in X : (x, x) \in U\}$ is analytic. If $A \in \mathbf{\Delta^1_1}$, then $X \setminus A \in \mathbf{\Sigma^1_1}$ and thus, there is some $x \in X$ such that $U_x = X \setminus A$. But this is impossible, since :

$$x \in A \text{ if and only if } (x, x) \in U \text{ if and only if } x \in X \setminus A$$

$\blacksquare$

It is worth to mention that using very similar arguments, one can actually prove the following analogous of Theorem 1.8 :

Theorem 1.25. *Let X be a Polish space. Then :*

(i) For all $n \geq 1$, there is a ω^ω-universal set $U \in \Gamma_n(\omega^\omega \times X)$ for $\Gamma_n(X)$, with $\Gamma_n = \boldsymbol{\Sigma}^1_n$ or $\Gamma_n = \boldsymbol{\Pi}^1_n$.

(ii) If X is uncountable, then there is some $U \in \Gamma_n(X \times X)$ which is universal for $\Gamma_n(X)$ and thus, $\boldsymbol{\Sigma}^1_n \neq \boldsymbol{\Pi}^1_n$.

Proof. The reader can find a proof in [2] (Lemma 4.1.10, p. 131 and Theorem 4.1.11, p. 132). ∎

1.3 Regularity Properties

1.3.1 Perfect Set Property

We start by introducing a standard technique from classical Descriptive Set Theory : schemes. We already used schemes in Proposition 1.5. Furthermore, many standard results on Polish spaces that we have been using, can be proven by using schemes.

Definition 1.26. Given any set X, a **Suslin scheme** is a family of subsets of X indexed by $\omega^{<\omega}$, $\{F_s\}_{s \in \omega^{<\omega}}$. If whenever $s \subseteq t$ one has that $F_t \subseteq F_s$, the Suslin scheme is said to be regular. Given a Suslin scheme, the **Suslin operation** is defined to be :

$$\mathcal{A}(\{F_s\}) := \bigcup_{\alpha \in \omega^\omega} \bigcap_n F_{\alpha|_n}$$

Proposition 1.27. The Suslin operation is idempotent.

Proof. The reader can find a proof in [2] (Theorem 1.13.1, p.35). ∎

In this section, we study the Perfect Set Property (and some cardinality related results) on analytic and coanalytic sets. We recall the definition of a perfect set :

Definition 1.28. A subset of a topological space is said to be perfect if it is closed and without isolated points.

We are now ready to introduce the PSP :

Definition 1.29. Let X be a Polish space and $A \subset X$. Then, A is said to have the perfect set property (PSP) if it is either countable or contains a non-empty perfect set.

Remark 1.30. Note that not all subsets of $\mathbb{R}$ have the PSP. Indeed, Bernstein sets (see Definition 4.18 and Proposition 4.19) are examples of uncountable sets which do not contain any perfect set.

Note that by Theorem 4.63, the Continuum Hypothesis (CH) holds for subsets of a Polish space which have the PSP. In Theorem 1.33, we prove that all analytic sets have the PSP (in particular, all Borel sets of a Polish space have the PSP), **proving in ZFC that the Continuum Hypothesis holds** for this class of sets. In order to do so, we introduce a particular kind of scheme :

Definition 1.31. Let X be a Polish space and A be any non-empty set endowed with the discrete topology. Consider a family $\{F_s\}_{s\in A^{<\omega}}$ of subsets of X such that :

$$\overline{F_{s^\frown a}} \subseteq F_s, \text{ for all } s \in A^{<\omega} \text{ and } a \in A$$

$$\text{diam}(F_{\alpha|_n}) \to 0, \text{ as } n \to \infty \text{ for all } \alpha \in A^\omega$$

$$F_s \cap F_t = \emptyset, \text{ for all } s \perp t$$

Then, $\{F_s\}_{s\in A^{<\omega}}$ is called a **Lusin scheme**.

Remark 1.32. Let $\{F_s\}_{s\in A^{<\omega}}$ be a Lusin scheme. Note that :

$$C = \{\alpha \in A^\omega : \forall n : F_{\alpha|_n} \neq \emptyset\}$$

is a closed subset of A^ω. Moreover, $f : C \to X$ such that :

$$f(\alpha) = \bigcap_n F_{\alpha|_n}$$

is a well defined continuous map which is injective. In particular, if $A = 2$, $\{F_s\}_{s\in 2^{<\omega}}$ is called a Cantor scheme and it follows that f is an embedding of 2^ω into X.

Theorem 1.33. *Every uncountable analytic set of a Polish space contains a homeomorphic copy of the Cantor set. Hence, all analytic sets have the PSP and in particular, the CH holds for analytic sets.*

Proof. Let X be a Polish space and $f : \omega^\omega \to X$ be a continuous map such that $f(\omega^\omega) \subset X$ is uncountable. We show that there is a Cantor scheme $\{F_s\}_{s\in 2^{<\omega}}$ of closed subsets of ω^ω such that whenever $|s| = |t|$ and $s \neq t$, then $f(F_s)$ and $f(F_t)$ are disjoint. It follows that $\mathcal{A}(\{F_s\}) := C \approx 2^\omega$. Moreover, $f|_C$ is injective : indeed, let $x \in \bigcap_n F_{\alpha|_n}$ and $y \in \bigcap_n F_{\beta|_n}$ such that $x \neq y$. Then, clearly $\alpha \neq \beta$ in 2^ω, so let k be such that $\alpha(k) \neq \beta(k)$. But $f(x) \in f(F_{\alpha|_k})$ and $f(y) \in f(F_{\beta|_k})$ and we conclude that $f(x) \neq f(y)$.
It follows that f is an embedding of 2^ω into $f(C)$ and we are done. Hence, it remains to show that such Cantor scheme exists.

Since the range of f is uncountable, let $Z \subseteq \omega^\omega$ be an uncountable set such that $f|_Z$ is one to one. By Theorem 4.64, we can assume without loss of generality that Z is dense in itself. We first define a family of non-empty open sets of ω^ω, $\{U_s : s \in 2^{<\omega}\}$, such that :

(i) diam(U_s)$< 2^{-|s|}$, for all $s \in 2^{<\omega}$

(ii) $U_s \cap Z \neq \emptyset$, for all $s \in 2^{<\omega}$

(iii) $\overline{U_{s\frown\epsilon}} \subseteq U_s$, for all $s \in 2^{<\omega}$ and $\epsilon \in \{0,1\}$

(iv) Whenever $|s| = |t|$ and $s \neq t$, for $s,t \in 2^{<\omega}$, then $f(\overline{U_s}) \cap f(\overline{U_t}) = \emptyset$

We then take $F_s := \overline{U_s}$ and it is clear that $\{F_s : s \in 2^{<\omega}\}$ is a Cantor scheme with the desired properties. It remains to define each U_s and we do it inductively on the length of $s \in 2^{<\omega}$:

We let $U_\emptyset = X$ and we suppose that U_s is defined for some s. Since Z is dense in itself, $U_s \cap Z$ has at least two points, say $x_0 \neq x_1$. Since $f|_Z$ is one to one, $f(x_0) \neq f(x_1)$ and there are disjoint open sets such that $f(x_0) \in W_0$ and $f(x_1) \in W_1$. Now, since f is continuous, there are $U_{s\frown 0}$ and $U_{s\frown 1}$ such that $f(\overline{U_{s\frown i}}) \subseteq W_i$ (for $i \in 2$) and using regularity of metric spaces, we choose each $U_{s\frown i}$ in a way such that properties (i)-(iv) are verified. ■

Remark 1.34. We just proved that analytic sets have the PSP. The statement of whether or not the PSP holds for coanalytic sets is **independent** from ZFC. Indeed, under $V = L$, Godel proved that there is an uncountable coanalytic subset that does not contain any perfect set (c.f [2], p.147). On the other hand, as we will see in Section 2.4, under the additional axiom of $\mathbf{\Sigma^1_1}$-determinacy, all coanalytic sets have the PSP.

Remark 1.35. A consequence of Theorem 1.33 is that the CH holds for analytic sets. The statement of whether or not the CH holds for coanalytic sets is **independent** from ZFC. However, we can prove in ZFC that an uncountable coanalytic set either has cardinality $\aleph_1$ or $\mathfrak{c}$. Indeed, this is the content of Corollary 1.40.

In order to establish the above result on the cardinality of coanalytic sets, we will rely on some basic results on trees. If needed, the reader can find the basic definitions (such as the definition of well and ill founded trees and ranks) in section 4.3, which provides a short and self-contained introduction to trees. Henceforth, we will assume that the standard terminology regarding trees is known. In particular, we denote the rank of an element of a tree T, $\sigma \in T$, by $\rho_T(\sigma)$.

We define sections of a tree (on a product) and then, we prove a classic - yet very useful - characterization of the (co)analytic sets of ω^ω in terms of sections of (well) or ill-founded trees on $\omega \times \omega$.

Suppose that T is a tree on $A \times B$, for any non-empty sets A and B and let $\alpha \in A^\omega$. We define the **section** of T at α by :

$$T(\alpha) := \{\beta \in B^{<\omega} : (\alpha|_{|\beta|}, \beta) \in T\}$$

It is clear that, if $\pi : A^\omega \times B^\omega \to A^\omega$ is the projection map :

$$\alpha \in \pi([T]) \text{ if and only if } T(\alpha) \text{ is ill-founded}$$

We can then prove the following characterization :

Proposition 1.36. Let $A \subseteq \omega^\omega$. The following are equivalent :

(i) A is coanalytic

(ii) There is a tree T on $\omega \times \omega$ such that $\alpha \in A$ if and only if $T(\alpha)$ is well-founded.

Proof. Let $A \subseteq \omega^\omega$ be a coanalytic set. Then, by Theorem 1.13 there is a closed subset $C \subseteq \omega^\omega \times \omega^\omega$ such that $\pi(C) = \omega^\omega \setminus A$. By Theorem 4.79, there is a tree T on $\omega \times \omega$ such that $[T] = C$. Thus, $\alpha \in \omega^\omega \setminus A$ if and only if there is some β such that $(\alpha, \beta) \in [T]$, which is equivalent to say that $T(\alpha)$ is ill-founded. Clearly, the converse follows similarly. ∎

Corollary 1.37. Let $A \subseteq \omega^\omega$. Then, the following are equivalent :

(i) A is analytic

(ii) There is a tree T on $\omega \times \omega$ such that $\alpha \in A$ if and only if $T(\alpha)$ is ill-founded.

Relying on the previous elegant characterization of (co)analytic subsets of ω^ω and on reduction theorems for Polish spaces, we can establish our cardinality result for coanalytic sets. It is worth to emphatize the central role of the space ω^ω :

Theorem 1.38. *Let X be a Polish space and $A \subseteq X$ which is analytic. Then, $A = \bigcap_{\alpha < \omega_1} B_\alpha$, with each B_α a Borel set.*

Proof. We note that by Theorem 4.74 it is enough to consider $X = \omega^\omega$. By Corollary 1.37, let T be a tree on $\omega \times \omega$ such that $A = \pi([T])$ and let $C = \omega^\omega \setminus A$. For $\alpha < \omega_1$ and $s \in \omega^{<\omega}$, define :

$$C_s^\alpha := \{x \in \omega^\omega : \rho_{T(x)}(s) \leq \alpha\}$$

Note that $x \in C$ if and only if there is some $\alpha < \omega_1$ such that $\rho_{T(x)}(\emptyset) \leq \alpha$. Therefore, letting $C_\alpha := C_\emptyset^\alpha$ it follows that $C = \bigcup_{\alpha < \omega_1} C_\alpha$.
Hence, if we prove that each C_α is Borel, it is enough to take $B_\alpha := \omega^\omega \setminus C_\alpha$ and we are done. We prove, by induction on $\alpha < \omega_1$, that each C_s^α is Borel. Note that :

$$\begin{aligned} C_s^0 &= \{x \in \omega^\omega : s \in T(x) \text{ is terminal or } s \notin T(x)\} \\ &= \{x \in \omega^\omega : \forall n (x|_{|s|+1}, s \frown n) \notin T\} \end{aligned}$$

Hence, C_s^0 is closed and the result follows by induction, noting that for $\alpha > 0$:

$$C_s^\alpha = \bigcap_n \bigcup_{\xi<\alpha} C_{s\frown n}^\xi$$

■

Corollary 1.39. Let X be a Polish space and $A \subseteq X$ which is coanalytic. Then, $A = \bigcup_{\alpha<\omega_1} B_\alpha$, with each B_α a Borel set.

Proof. While the result follows directly from Theorem 1.38, it is worth to note that the construction of such Borel sets was already defined in the proof of the aforementioned theorem. ■

Corollary 1.40. Let X be a Polish space and $A \subseteq X$ be coanalytic. Then, either A is countable, has cardinality $\aleph_1$ or $\mathfrak{c}$.

Proof. Since any Borel set is analytic, it follows by Theorem 1.33 that it is either countable or has cardinality $\mathfrak{c}$. The result follows immediately by Corollary 1.39 and Theorem 4.38. ■

1.3.2 Baire Property and Lebesgue measurability

The next two properties that we consider are the Baire Property (BP) and Lebesgue measurability (LM). We establish that analytic sets satisfy both the BP and LM (and thus, since both the class of sets which satisfy the BP and the class of LM sets are closed under complements, coanalytic sets also satisfy the BP and are Lebesgue measurable).

First, let's recall some facts about the Baire Property.

Proposition 1.41. Let X be any topological space and $A \subseteq X$. Then, the following are equivalent :

(i) There is some open set $\mathcal{U} \subseteq X$ such that $A\Delta\mathcal{U}$ is meager.

(ii) $A = G \cup M$, for some G_δ-set G and meager set M.

(iii) $A = F \setminus M$, for some F_σ-set F and meager set M.

Proof. The reader can find a proof in [1] (Proposition 8.23, p.48). ■

For a topological space X and $A \subseteq X$, A is said to have the **Baire Property** (BP) if it satisfies one (and thus, all) of the above conditions.

Proposition 1.42. Let X be a topological space. Then, the set of sets that have the BP is a σ-algebra. In particular, every Borel set has the BP.

Proof. The reader can find a proof in [1] (Proposition 8.22, p.49). ■

Proposition 1.43. Let X be a Polish space and $A \subseteq X$. Then, there is some $B \subseteq X$ such that $A \subseteq B$, B has the BP and such that if $D \subseteq B \setminus A$ has the BP, then D is meager.

Proof. Let $\{\mathcal{U}_n\}$ be a countable basis for X and define :

$$A^* = \{x \in X : \forall i(x \in \mathcal{U}_i \Rightarrow \mathcal{U}_i \cap A \text{ is non meager})\}$$

It follows by definition that $A \setminus A^* = \bigcup\{A \cap \mathcal{U}_i : \; A \cap \mathcal{U}_i \text{ is meager}\}$ is meager, since it is a countable union of meager sets. Moreover, A^* is closed : if $x \notin A^*$ then there is some i such that $x \in \mathcal{U}_i$ and $\mathcal{U}_i \cap A$ is meager. If $y \in \mathcal{U}_i$, then $y \notin A^*$, otherwise $\mathcal{U}_i \cap A$ would be non meager. Thus, $\mathcal{U}_i \subseteq X \setminus A^*$ and we conclude that A^* is closed. Then, we let :

$$B := A \cup A^* = A^* \cup (A \setminus A^*)$$

Note that B has the BP since it is the union of a meager set and a closed set. It remains to prove that if $D \subseteq B \setminus A$ is such that has the BP, then D is meager. In order to do so, it is enough to prove that if $B' \supseteq A$ has the BP, then $C = B \setminus B'$ is meager. Indeed, suppose towards a contradiction that C is non meager. Since C has the BP, there is some i such that $\mathcal{U}_i \neq \emptyset$ and $\mathcal{U}_i \setminus C$ is meager. It follows that $\mathcal{U}_i \cap A \subseteq \mathcal{U}_i \setminus C$ is also meager. Since $\mathcal{U}_i \cap C \neq \emptyset$ (otherwise $\mathcal{U}_i$ is meager, contradicting the Baire Category Theorem) it follows that there is some $x \in \mathcal{U}_i$ such that $x \notin A^*$. Consequently, $\mathcal{U}_i \cap A$ would be non meager, yielding to a contradiction. ∎

Remark 1.44. The analogous of Proposition 1.43 for Lebesgue measurable sets is true. More concretely, if $A \subseteq \mathbb{R}$ then there is some $B \subseteq \mathbb{R}$ such that $A \subseteq B$, B is Lebesgue measurable and such that if $D \subseteq \mathbb{R}$ is Lebesgue measurable and $D \subseteq B \setminus A$, then D is null. The reader can find the proof of this standard fact in Claim 1 (Theorem 2.2).

It is worth to recall that there are more analogies between Lebesgue measurable sets and sets that have the BP and between null and meager sets :

Fact 1 : In ZFC, every subset of $\mathbb{R}$ which is not null, contains a non measurable subset. Analogously, given a topological space X, any $A \subseteq X$ which has the BP and is not meager, contains a subset which does not have the BP.

Fact 2 : In ZFC, Bernstein sets are examples of subsets of $\mathbb{R}$ which are not Lebesgue measurable and do not have the Baire Property.

Fact 3 : Let us now focus on $X = \mathbb{R}$. Then, there are sets with the BP which are not Lebesgue measurable and there are Lebesgue measurable sets without the BP. Indeed, for an example of a set which is meager and non measurable, it is enough to consider any set which is meager and not null. Since every subset of a meager set is meager, the result follows from Fact 1. We consider $\{q_n\}$ to be an enumeration of $\mathbb{Q} \cap [0,1]$ and for $\epsilon > 0$ sufficiently small, let:

$$A_\epsilon = \bigcup_n (q_n - \tfrac{\epsilon}{2^n}, q_n + \tfrac{\epsilon}{2^n}) \subseteq [0,1]$$

Then, $\mu(A_\epsilon) \leq \epsilon$ and thus, $\mu([0,1] \setminus A_\epsilon) \geq 1 - \epsilon$. Moreover, note that $[0,1] \setminus A_\epsilon$ is nowhere dese. For each n, we can adjust $\epsilon(n)$ so that $\mu([0,1] \setminus A_{\epsilon(n)}) \geq 1 - \frac{1}{n}$. It follows that $C := \bigcup_n [0,1] \setminus A_{\epsilon(n)}$ is meager and such that $\mu(C) = 1$. On the other hand, to prove that there is a Lebesgue measurable set which does not have the BP, it is enough to prove that there is a set which is Lebesgue measurable, null, with the BP and which is not meager. Again, the result will follows from Fact 1. We simply note that we already know how to construct open sets $\mathcal{U}_n$ such that contain all rationals and $\mu(\mathcal{U}_n) \leq \frac{1}{n}$. Then, we just take $\mathcal{U} = \bigcap_n \mathcal{U}_n$ which is not meager by the Baire Category Theorem.

We prove now an important representation theorem of analytic sets. This shall be used in order to establish more regularity properties of analytic sets, such as the BP and Lebesgue measurability.

Theorem 1.45. *Let X be a Polish space and $A \subseteq X$. Then, the following are equivalent :*

(i) A is analytic

(ii) There is a regular Suslin scheme of closed sets $\{F_s\}_{s \in \omega^{<\omega}}$ such that $A = \mathcal{A}(\{F_s\})$.

(iii) There is a Suslin scheme of closed subsets $\{F_s\}_{s \in \omega^{<\omega}}$ such that $A = \mathcal{A}(\{F_s\})$.

Proof. Since $(ii) \Rightarrow (iii)$ trivially, it is enough to prove that :

$(i) \Rightarrow (ii)$: Let $A \subseteq X$ be analytic and let $f : \omega^\omega \to X$ be a continuous map such that $f(\omega^\omega) = A$. For any $s \in \omega^{<\omega}$, take $F_s := \overline{f(\Sigma(s))}$. Clearly, $A \subseteq \mathcal{A}(\{F_s\})$ since if $a \in A = f(\omega^\omega)$ then there is some $\alpha \in \omega^\omega$ such that for all $n \in \omega$ one has that $a \in F_{\alpha|_n}$. Conversely, let $x \in \mathcal{A}(\{F_s\})$ so that there is some $\alpha \in \omega^\omega$ such that for all $n \in \omega$ one has that $x \in \overline{f(\Sigma(\alpha|_n))}$. Hence, for all $n \in \omega$ there is some $\alpha_n \in \Sigma(\alpha|_n)$ such that $f(\alpha_n) \to x$. It follows, using continuity of f, that $x = f(\alpha)$ with $\alpha = \lim_n \alpha_n$ and thus, $x \in A$. Thus, we proved that $\mathcal{A}(\{F_s\}) = A$. It remains to prove that :

$(iii) \Rightarrow (i)$: Now, we assume that we have a family of closed subsets of X, $\{F_s\}_{s \in \omega^{<\omega}}$, and we take $A = \mathcal{A}(\{F_s\})$. In other words, $x \in A$ if and only if there is some $\alpha \in \omega^\omega$ such that for every n one has that $x \in F_{\alpha|_n}$. It is enough to define a Borel set on $X \times \omega^\omega$ such that its projection is A. We define :

$$B := \{(x, \alpha) : \forall n (x \in F_{\alpha|_n})\}$$

It is immediate that A is the projection of B so it remains to check that B is Borel. This follows from the following equality, since each $F_s \times \Sigma(s)$ is closed:

$$B = \bigcap_n \bigcup_{s:|s|=n} (F_s \times \Sigma(s))$$

■

Proposition 1.46. The class of Lebesgue measurable subsets of $\mathbb{R}$ is closed under the Suslin operation.

Proof. Let $\{A_s\}_{s\in\omega^{<\omega}}$ be a Suslin scheme of Lebesgue measurable sets. By Theorem 1.45 one can assume it is a regular scheme. For each $s \in \omega^{<\omega}$, define :

$$A^s = \bigcup_{\alpha \supseteq s} \bigcap_{n\in\omega} A_{\alpha|_n} \subseteq A_s$$

Note that in particular, $A^\emptyset = \mathcal{A}(\{A_s\})$. Moreover, for each $s \in \omega^{<\omega}$ let $B^s \supseteq A^s$ be as in Remark 1.44. Considering $B^s \cap A_s$ we can assume without loss of generality that $B^s \subseteq A_s$ and thus, $\{B^s\}$ is also a regular Suslin scheme. We will define a null set C such that $B^\emptyset \setminus C \subseteq A^\emptyset$. Since $B^\emptyset \setminus A^\emptyset \subseteq C$, this implies that $B^\emptyset \setminus A^\emptyset$ is null and thus, $A^\emptyset = \mathcal{A}(\{A_s\})$ is Lebesgue measurable.
Let $C_s = B^s \setminus \bigcup_n B^{s\frown n}$. Since $\bigcup_n B^{s\frown n}$ is measurable and $A^s \subseteq \bigcup_n B^{s\frown n}$ it follows, by choice of B^s, the each C_s is null and thus, $C = \bigcup_s C_s$ is also null.
It remains to check that $B^\emptyset \setminus C \subseteq A^\emptyset$. Indeed, let $x \in B^\emptyset \setminus C$. Since $x \notin C_\emptyset$, there is some $\alpha(0)$ such that $x \in B^{\alpha(0)}$. Assuming by induction, that $x \in B^{\alpha|_n}$, and since $x \notin C_{\alpha|_n}$, there is some $\alpha(n)$ such that $x \in B^{\alpha|_{n+1}}$ and thus, we can construct an element $\alpha \in \omega^\omega$ such that $x \in \bigcap_n B^{\alpha|_n} \subseteq \bigcap_n A_{\alpha|_n} \subseteq A^\emptyset$. ■

Proposition 1.47. Let X be any topological space. Then, the class of subsets of X which have the BP, is closed under the Suslin operation.

Proof. Since the family of sets which have the BP is a σ-algebra, one can easily adapt the proof of Proposition 1.46, using Proposition 1.43. ■

Corollary 1.48. Let X be any Polish space. Then, all analytic and coanalytic subsets of X have the BP.

Proof. It follows from Theorem 1.45 and Proposition 1.47. ■

Corollary 1.49. All analytic and coanalytic subsets of $\mathbb{R}$ are Lebesgue measurable.[1]

Proof. It follows from Theorem 1.45 and Proposition 1.46. ■

Remark 1.50. It should be noted that the statement of whether or not all $\mathbf{\Delta^1_2}$ sets have the BP is **independent** from ZFC. The analogous statement for Lebesgue measurability of all sets of the class $\mathbf{\Delta^1_2}$ is also **independent** from ZFC. However, under some game-theoretic axioms (for instance, under Projective Determinacy and the Axiom of Determinacy - respectively for the BP and for Lebesgue measurability) these questions have an affirmative answer. We will see that this is the case in the next section (more concretely in 2.1 and 2.4).

[1] Actually, this result can be dramatically generalized : every analytic subset of a Polish space is universally measurable.

2 Determinacy

2.1 Introduction

In this section we will introduce infinite topological games with perfect information. The goal is twofold : in a sense, the *complexity* of a subset of a Polish space is related with its *determinacy* in ZFC. On the other hand, we mentioned previously that the statement of whether or not the CH holds for coanalytic sets, is independent from ZFC. In this spirit, we introduce the axiom of $\mathbf{\Sigma^1_1}$-determinacy and study how the determinacy of a certain game implies the PSP for coanalytic sets.

We begin by describing the following game, where A is a discrete space, A^ω has the product topology and $X \subseteq A^\omega$:

Player	Moves			
P1	a_0		a_2	
P2		a_1		a_3

There are two players, P1 and P2. At each turn, a player chooses an element $a_i \in A$ and we assume that this process goes *ad infinitum* so that in the end we have some $x = (a_i)_{i\geq 0} \in A^\omega$. We say that P1 wins this game if and only if $x \in X$, so X is our *payoff set*. Further, we refine this definition with the notion of game with rules. For now, we denote this game by $G(A, X)$.

A strategy for P1 is a map $\sigma : \{s \in A^{<\omega} : |s| \text{ is even}\} \to A$. Intuitively, and as the name suggests, a strategy for P1 is simply a rule that tells what P1 should play when it is its turn to make a move and this choice depends on what both players played before. Similarly, a strategy for P2 is a map $\tau : \{s \in A^{<\omega} : |s| \text{ is odd}\} \to A$.

Now let σ be a strategy for P1 and $y = (y_0, y_1, ...) \in A^\omega$. We define $\sigma(y) = (x_0, y_0, x_1, y_1, ...)$ to be the *play in this game where x_i are choices of P1 by following σ if P2 plays y_i at each turn*. More formally :

(i) $x_0 = \sigma(\emptyset)$

(ii) $x_{n+1} = \sigma((x_0, y_0, ..., x_n, y_n))$

Then, if σ is a strategy for P1, we define $[\sigma] := \{\sigma(y) : y \in A^\omega\}$. Similarly, we have analogous definitions for a strategy τ for P2.
We say that σ is a **winning strategy** for P1 if $[\sigma] \subseteq X$ and similarly, τ is a winning strategy for P2 if $[\tau] \cap X = \emptyset$. Finally, we say that a game $G(A, X)$ is **determined** (or simply the set $X \subseteq A^\omega$ is determined) if one of the players has a winning strategy. For instance, it is easy to check that if X is finite, then X is determined. It is also easy to prove that if X is open or closed, then X

is determined (see Theorem 2.3). We will see in section 2.2 that actually, if X is Borel then X is determined and this is, in a sense, the best we can get in ZFC.[2]

It should be noted that under ZFC and if, for instance $A = \omega$, then there are games which are not determined :

Proposition 2.1. (ZFC) There is a game $G(\omega, P)$ which is not determined.

Proof. Note that there are $2^{\aleph_0}$ strategies for P1 (and also, for P2). Using the Axiom of Choice, let's consider $\{\sigma_\alpha\}_{\alpha \in I}$ and $\{\tau_\alpha\}_{\alpha \in I}$ to be the list of strategies, respectively for P1 and P2, indexed by some well-ordered set I with cardinality $2^{\aleph_0}$. We define two disjoint sets P and Q by transfinite recursion. At first, pick $p_0 \in [\tau_0]$ and $q_0 \in [\sigma_0]$ such that $p_0 \neq q_0$. Let $\alpha \in I$ and suppose that for every $\beta < \alpha$ we have already picked p_β and q_β such that $p_\beta \neq q_\beta$ and $p_\beta \in [\tau_\beta], q_\beta \in [\sigma_\beta]$. At the α^{th} stage of the construction, pick some $p_\alpha \notin \{q_\beta\}_{\beta<\alpha}$ and some $q_\alpha \notin \{p_\beta\}_{\beta<\alpha} \cup \{p_\alpha\}$ such that $p_\alpha \in [\tau_\alpha]$ and $q_\alpha \in [\sigma_\alpha]$.
Then, let $P = \{p_\alpha\}_{\alpha \in I}$ and $Q = \{q_\alpha\}_{\alpha \in I}$. It follows by construction that $P \cap Q = \emptyset$. Moreover, $G(\omega, P)$ is not determined : indeed, suppose that P1 has a winning strategy, $\sigma = \sigma_\alpha$. Since $q_\alpha \in Q$ it follows that $[\sigma_\alpha] \subsetneq P$ and thus, σ is not winning. Similarly for P2, so we conclude that $G(\omega, P)$ is not determined. ■

To summarize, Proposition 2.1 implies that under ZFC not every subset $X \subseteq \omega^\omega$ is determined. However, as we will see in Theorem 2.4, every $X \subseteq \omega^\omega$ which is Borel is also determined. In a sense, this is the *most general* class of sets which are determined if we work within ZFC (this will be discussed in section 2.4). There are, however, alternatives to ZFC and in this context we introduce the Axiom of Determinacy.

Axiom of Determinacy (AD) : Every game $G(\omega, X)$ is determined.

Clearly, by Proposition 2.1, AD is not consistent with the Axiom of Choice. Furthermore, as we will discuss in section 2.4, the AD implies that every subset of $\mathbb{R}$ has the PSP and the BP. In that section, we will see how games can be used to study topological spaces and properties of pointclasses.

Another regularity property discussed in section 1.3 was Lebesgue measurability. In what follows, we prove that under AD (and within ZF+AD) every subset of the real line is Lebesgue measurable. Note that this shows, once again, that AD is not consistent with the AC.[3] We should comment that even though the use of the word *strategy* is hopefuly clear, its definition in this proof is perhaps better formalized in the context of games with rules (section 2.2).

[2]If we consider the Borel and projective hierarchies, then $\boldsymbol{\Delta}^1_1$ is the *highest* hierarchy for which determinacy is assured in ZFC. As we will see, determinacy for $\boldsymbol{\Sigma}^1_1$ can't be proven (or disproven) in ZFC.

[3]Recall that under ZFC, there are non measurable sets. For instance, a Vitali or a Bernstein set. There are models of ZF where every subset of the real line is measurable, which shows the nonconstructive aspect of these pathological sets that *require choice* in order to be defined.

Theorem 2.2. *(AD) Every subset $A \subseteq \mathbb{R}$ is Lebesgue measurable.*

Proof. Note that throughout this argument, we work within ZF. Assuming the following claims, the result follows immediately (note that Claim 2 is false in ZFC, as a Bernstein set provides a counter-example) :

Claim 1 : Let $T \subseteq \mathbb{R}$. Then, there is some $G \in \mathcal{L}(\mathbb{R})$ such that $T \subseteq G$, $\mu^*(G) = \mu^*(T)$ and if $N \subseteq G \setminus T$ is measurable, then N is null.[4]

Claim 2 (AD) : If $S \subseteq \mathbb{R}$ is such that every $N \subseteq S$ which is measurable is null, then S is null.[5]

Indeed, let's assume both claims and consider any $A \subseteq \mathbb{R}$. Let $A \subseteq G$, with G as in Claim 1. Note that $A = G \setminus (G \setminus A)$ and that, by Claim 2, $G \setminus A$ is Lebesgue measurable. It follows that A is Lebesgue measurable and thus, it is enough to prove both claims :

Proof of Claim 1 : First, consider the case when T is bounded so that we have $\mu^*(T) < \infty$. Thus, for each n, there is a collection of basic open intervals, say $\{I^n_k\}_{k\in\omega}$, such that $T \subseteq \bigcup_k I^n_k$ and $\mu^*(\bigcup_k I^n_k) \leq \mu^*(T) + \frac{1}{n}$. We simply take $G = \bigcap_n \bigcup_k I^n_k$. If T is unbounded, we note that $T = \bigcup_n T_n$ with $T_n = T \cap [-n, n]$. But each T_n is bounded and thus, there exists some $T_n \subseteq G_n$ as in the previous case. We simply take $G = \bigcup_n G_n$.

Proof of Claim 2 : Without loss of generality we can assume that $S \subseteq [0, 1]$. For a sequence $(a_i)_{i\in\omega} \in 2^\omega$, we define a real number $\varphi(a_i) := \sum_i \frac{a_i}{2^{i+1}} \in [0, 1]$. For a fixed $\epsilon > 0$ and each $n \in \omega$, we consider the collection $\mathcal{C}_n = \{G^n_k\}_{k\in\omega}$ of all finite unions G^n_k of intervals with rational endpoints such that $\mu^*(G^n_k) \leq \frac{\epsilon}{2^{2n+1}}$. Finally, we describe a certain game (usually called a *covering game*) which, under the AD, is determined :

Consider two players, P1 and P2, choosing at each turn an element $n \in \omega$. For the sake of readability, suppose P1 chooses some $a_i \in \omega$ at each turn and that P2 chooses some $b_i \in \omega$ at each turn so that one has the sequence $(a_0, b_0, a_1, b_1, ...) \in \omega^\omega$. The covering game is won by P1 if and only if each $a_i \in \{0, 1\}$, $\varphi(a_i) \in S$ and if $\varphi(a_i) \notin \bigcup_n G^n_{b_n}$. Otherwise, P2 wins.

We claim that P1 can't have a winning strategy. Indeed, suppose that σ is a winning strategy for P1. We can define $f : \omega^\omega \to \mathbb{R}$ by $f((b_i)_{i\geq 0}) := a$, with $a = \sum_i \frac{a_i}{2^{i+1}}$, where a_i is played by P1 following σ in response to P2 playing with each b_i at each turn. Clearly, f is continuous and since σ is winning, $f(\omega^\omega) \subseteq S$. Furthermore, as $f(\omega^\omega)$ is analytic, it follows by Corollary 1.49 and

[4]In this context, μ^* is the Lebesgue outer measure.

[5]This is false in ZFC : Let B be a Bernstein set. Since B does not contain any uncountable closed set, it follows that each measurable $N \subseteq B$ is null. However, B is not null since every open set which contains B, has full measure.

by assumption that $f(\omega^\omega)$ is null. Thus, $f(\omega^\omega) \subseteq \bigcup_n G^n_{b_n}$ for some $(b_0, b_1, ...)$ which then can be played by P2, contradicting the assumption that σ was a winning strategy.

Hence, by the AD, P2 has a winning strategy τ. For each $s = (a_i)_{i=1}^n \in 2^{<\omega}$ let $G_s := G^n_{b_n} \in \mathcal{C}_n$, where b_n is such that $\tau((a_0, b_0, ..., a_n)) = b_n$. Since τ is a winning strategy for P2, $S \subseteq \bigcup_n \bigcup_{s \in 2^n} G_s$ from where it follows that $\mu^*(S) \leq \epsilon$ and thus, S is null. ■

2.2 Borel Determinacy

It is convenient to introduce the notion of *game with rules*. Given a discrete set A we consider a tree T on A which determines the legal positions in a certain game. Indeed, let T be a non-empty pruned tree and let $X \subseteq [T] \subseteq A^\omega$. There are two players, P1 and P2, that at each turn pick some element from A in a way such that for each n, the sequence of elements picked by both players is such that $(a_0, a_1, ..., a_n) \in T$. We then say that P1 wins this game if and only if $x = (a_i)_{i \geq 0} \in X$. Otherwise, P2 wins the game. We denote this game by $G(T, X)$ and we define strategy and winning strategy for P1 (or P2) in a completely analogous way as we did in the previous section. For instance, a strategy σ for P1 is a non-empty pruned subtree $\sigma \subseteq T$ such that :

(i) If $(a_0, ..., a_{2n}) \in \sigma$, then for all a_{2n+1} such that $(a_0, ..., a_{2n+1}) \in T$, we have that $(a_0, ..., a_{2n+1}) \in \sigma$

(ii) If $(a_0, ..., a_{2n-1}) \in \sigma$, then there is an unique a_{2n} such that $(a_0, ..., a_{2n}) \in \sigma$

(iii) We say that σ is winning if $[\sigma] \subseteq X$

It should be noted that we are not considering *a larger class* of games, relatively to what we defined in the previous section. Indeed, consider a game $G(T, X)$ for a tree T on A. We say that two games G and G' are equivalent whenever the existence of a winning strategy for a player in G is equivalent to the existence of a winning strategy for the same player in G'. We claim that $G(T, X)$ is equivalent to the game (using the notation of the previous section) $G(A, X')$ with the following payoff set :

$$X' = \{x \in A^\omega : [(\mathcal{B}_x \neq \emptyset) \wedge (\min \mathcal{B}_x \text{ is even})] \ \vee [(x \in [T]) \ \wedge (x \in X)]\}$$

$$\text{where } \mathcal{B}_x = \{n \in \omega : \exists s \in A^{<\omega} \text{ such that } s \notin T,\ s \subseteq x \text{ and } |s| = n\}$$

Indeed, suppose that σ is a winning strategy for P1 in $G(T, X)$, i.e $\sigma \subseteq T$ is a pruned subtree such that the above conditions (i)-(iii) hold. We now define a winning strategy σ' for P1 on $G(A, X')$: we let σ' request P1 to start the game $G(A, X')$ by playing the unique $a_0 \in A$ such that $(a_0) \in \sigma$. Then, suppose that P2 replies with a_1. If $(a_0, a_1) \in T$, then σ' requests P1 to answer with the unique $a_2 \in A$ such that $(a_0, a_1, a_2) \in \sigma$. Otherwise, if $(a_0, a_1) \notin T$, σ requests

P1 to always play a_0 (actually, it does not matter what P1 plays from now on, since any play x of this game, will be such that $x \in X'$). In this way, one defines σ' inductively for all moves : if $(a_0, ..., a_{2n-1}) \in T$, then σ' follows σ and if $(a_0, ..., a_{2n-1}) \notin T$, σ' makes P1 to play a_0 from then on. Similarly, a winning strategy for P2 in $G(T, X)$ induces a winning strategy for P2 in $G(A, X')$. On the other hand, a winning strategy in $G(A, X')$ translates to a winning strategy in $G(T, X)$.

So in essence, we are just reformulating our definitions of games within a more *appropriate* context.

Let T be a non-empty pruned tree on any set A and consider the game $G(T, X)$. Given a position of the game (node of T) $p = (a_0, ..., a_{2n-1})$, we say that it is **not losing** for P1 if P2 has no winning strategy on $G(T_p, X_p)$, where $T_p = \{s : p \frown s \in T\}$ and $X_p = \{x : p \frown x \in X\}$. Note that if p is not losing for P1 there is, by definition, some $(a_{2n}) \in T_p$ such that for any a_{2n+1} such that $(a_{2n}, a_{2n+1}) \in T_p$ then $p \frown (a_{2n}, a_{2n+1})$ is still not losing for $P1$.

We start with the following fundamental result :

Theorem 2.3. *Let T be a non-empty pruned tree on any set A and let $X \subseteq [T]$ be any open or closed subset. Then, $G(T, X)$ is determined.*

Proof. It is enough to suppose that X is closed. As it will be clear, if X is open we simply switch the roles of P1 and P2 in the argument of this proof.
Suppose that $P2$ has no winning strategy on $G(T, X)$ and note that $\emptyset$ is then a not losing position for P1. We define a strategy σ for P1 as follows : P1 chooses some $a_0 \in A$ such that $(a_0) \in T$ and such that for all a_1 such that $(a_0, a_1) \in T$, then (a_0, a_1) is still not losing for P1. In response, after P1 have played a_1, P2 will play some a_2 such that for any a_3 played by P2, $(a_0, ..., a_3) \in T$ is still not losing for P1 and so on. It is then clear that σ is a winning strategy for P1. Indeed, let $(a_n)_{n \geq 0}$ be a play of σ and suppose that $(a_n) \notin X$. Since $[T] \setminus X$ is open, there is some k such that $\Sigma((a_0, ..., a_{2k-1})) \cap [T] \subseteq [T] \setminus X$ which implies that $(a_0, ..., a_{2k-1})$ is not a not losing position for P1, contradicting the definition of σ. ∎

Let T be a non-empty pruned tree on some set A. A **quasistrategy** for P1 is simply a non-empty pruned tree $\mathcal{Q} \subseteq T$ such that if $(a_0, ..., a_{2j}) \in \mathcal{Q}$ and $(a_0, ..., a_{2j+1}) \in T$, then $(a_0, ..., a_{2j+1}) \in \mathcal{Q}$. So a quasistrategy is simply *a strategy without a necessarily unique decision.* We say that a quasistrategy $\mathcal{Q}$ is winning for P1 if $[\mathcal{Q}] \subseteq X$ and we define everything analogously for P2.

We defined what we mean by a not losing position $p = (a_0, ..., a_{2j-1})$ for P1. If on the other hand, $p = (a_0, ..., a_{2j})$, then we say that p is not losing for P1 if P2 has no winning strategy on $G(T_p, X_p)$ under the convention that P2 starts the game first. With this in mind, we define the **canonical quasistrategy** for P1 as : (and similarly for P2)

$$\mathcal{Q} = \{p \in T : \text{p is not losing for P1}\}$$

The main goal of this section is to prove the following :

Theorem 2.4. *Let T be a non-empty pruned tree on any set A and let $X \subseteq [T]$ be a Borel set. Then, $G(T, X)$ is determined.*

Theorem 2.3 establishes the determinacy of sets at the levels $\boldsymbol{\Sigma}_1^0$ and $\boldsymbol{\Pi}_1^0$. It was proven originally in [27]. In the following years, determinacy was proven for higher levels of the Borel hierarchy but it was only in 1975 that D. Martin (c.f [28]) proved that every Borel set is determined. In what remains from this section, we follow a *huge* simplification of the argument (c.f [5]) which also appears, for instance in [1] (Theorem 20.6).

In order to prove 2.4, we introduce the notions of k-covering and unravelling that will allow us to *simulate* the game $G(T, X)$ with an auxiliary game which is known to be determined by Theorem 2.3. We start with the definition of covering :

Definition 2.5. Let T be a non-empty pruned tree on any set A. A covering of T is a triple $(\tilde{T}, \pi, \varphi)$ such that :

(i) $\tilde{T}$ is a non-empty pruned tree on some set B

(ii) $\pi : \tilde{T} \to T$ is a monotone and length preserving map, which induces a natural continuous map from $[\tilde{T}]$ into $[T]$ that we also denote with π

(iii) φ maps strategies for P1 on $\tilde{T}$ (resp. P2) to strategies for P1 on T (resp. P2) in a way such that $\varphi(\sigma)$ restricted to positions of length $\leq n$ only depend on σ restricted to positions of length $\leq n$.

(iv) If $\tilde{\sigma}$ is a strategy for P1 on $\tilde{T}$ (resp. P2) and $x \in [\varphi(\tilde{\sigma})]$ then there is some $\tilde{x} \in [\tilde{\sigma}]$ such that $\pi(\tilde{x}) = x$.

As an example of a covering, consider the following tree T on ω :

$$T = \{\emptyset\} \cup \{a \in \omega^{<\omega} : a(0) = 0 \text{ and } a(j) \in \{0, 1\}, \text{ for } j \geq 1\}$$

Moreover, define $\tilde{T} \subseteq \omega^{<\omega}$ to be the following tree :

$$\tilde{T} = \{\emptyset\} \cup \{a \in \omega^{<\omega} : a(0) = 1 \text{ and } a(j) \in \{0, 1\}, \text{ for } j \geq 1\}$$

Let $\pi : \tilde{T} \to T$ such that $(a_0, ..., a_n) \mapsto (\varsigma(a_0), ..., \varsigma(a_n))$, where the map $\varsigma : \{0, 1\} \to \{0, 1\}$ is such that $\varsigma(0) = 1$ and $\varsigma(1) = 0$ and let φ map strategies on $\tilde{T}$ to strategies on T according to π. Then, $(\tilde{T}, \pi, \varphi)$ is a covering of T. Furthermore, let $X \subseteq [T]$ be the set $X = \{(0, 0, ...)\}$. It is clear that $\pi^{-1}(X)$ is a clopen set in $[\tilde{T}]$. We say that the covering unravels X :

Definition 2.6. A covering $(\tilde{T}, \pi, \varphi)$ unravels $X \subseteq [T]$ if $\pi^{-1}(X) := \tilde{X}$ is clopen in $[\tilde{T}]$.

Remark 2.7. Note that if $(\tilde{T}, \pi, \varphi)$ unravels $X \subseteq [T]$, then it also unravels $[T] \setminus X$.

Given a non-empty pruned tree T and a covering $(\tilde{T}, \pi, \varphi)$, one can *simulate* a game $G(T, X)$ by a game $G(\tilde{T}, \tilde{X})$, with $\tilde{X} = \pi^{-1}(X)$. Indeed, if $\tilde{\sigma}$ is a winning strategy for P1 in $G(\tilde{T}, \tilde{X})$, then $\varphi(\tilde{\sigma})$ is a winning strategy for P1 in $G(T, X)$. If that was not the case, then there is some $x \in [\varphi(\tilde{\sigma})]$ such that $x \notin X$. If we let $\tilde{x} \in [\tilde{\sigma}]$ such that $\pi(\tilde{x}) = x$, we get that $x \in X$. Similarly with P2. Hence, if $(\tilde{T}, \pi, \varphi)$ unravels $X \subseteq [T]$, we know by Theorem 2.3 that $G(\tilde{T}, \tilde{X})$ is determined and thus, so is the game $G(T, X)$.

Definition 2.8. For any $k \in \omega$ we say that $(\tilde{T}, \pi, \varphi)$ is a k-covering if it is a covering of T such that $T|_{2k} = \tilde{T}|_{2k}$ and $\pi|_{(\tilde{T}|_{2k})}$ is the identity map.

It is worth to comment that it may not be apparent the reason why we defined unravellings in terms of clopen sets, instead of merely open (or closed) sets. However, this definition will be useful in a transfinite argument that we use in the proof of Theorem 2.9. Furthermore, it may not be apparent the reason why we need to refine the definition of coverings and use k-coverings, but once again this is simply for technical reasons, due to the nature of the inductive argument that we will follow.

Note that in order to prove Theorem 2.4 it suffices to prove the following :

Theorem 2.9. *Let T be a non-empty pruned tree on a set A. Then, for every $k \in \omega$ and every $X \subseteq [T]$ which is Borel, there is a k-covering of T which unravels X.*

The proof of Theorem 2.9 heavily relies on the following two results, that we assume to be true for a moment :

Proposition 2.10. Let T be a non-empty pruned tree on a set A and let $X \subseteq [T]$ be closed. Then, for any $k \in \omega$, there is a k-covering of T that unravels X.

Proposition 2.11. Let $k \in \omega$ and $(T_{i+1}, \pi_{i+1}, \varphi_{i+1})$ be a $(k+i)$-covering of T_i, for $i \geq 0$. Then, there is a pruned tree T_∞ and maps $\pi_{\infty,i}$,$\varphi_{\infty,i}$ such that $(T_\infty, \pi_{\infty,i}, \varphi_{\infty,i})$ is a $(k+i)$-covering of T_i and such that $\pi_{i+1} \circ \pi_{\infty,i+1} = \pi_{\infty,i}$ and $\varphi_{\infty,i} = \varphi_{i+1} \circ \varphi_{\infty,i+1}$.

Essentially, Proposition 2.10 is the base case for the induction argument used in the proof of Theorem 2.9 and Proposition 2.11 establishes the existence of an object which can be thought of as an inverse limit. We now return to the proof of Theorem 2.9 :

Proof. (*of Theorem 2.9*) : Since X is Borel, $X \in \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^{\mathbf{0}}_{\alpha} = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^{\mathbf{0}}_{\alpha}$. We prove the result by induction on α. Proposition 2.10 proves the result for $\alpha = 1$,

since $\mathbf{\Pi^0_1}$ is the set of all closed subsets and if a given covering unravels $X \subseteq [T]$, then it unravels $[T] \setminus X$ as well.
We now suppose that $X \in \mathbf{\Sigma^0_\alpha}$, in which case $X = \bigcup_{i=0} X_i$ with $X_i \in \mathbf{\Pi^0_{\beta_i}}$, for $\beta_i < \alpha$. By induction hypothesis, there is a k-covering (T_1, π_1, φ_1) of $T = T_0$ which unravels X_0, i.e $\pi_1^{-1}(X_0)$ is clopen. Moreover, T_1 is a non-empty pruned tree by definition and we note that $\pi_1^{-1}(X_1) \in \mathbf{\Pi^0_{\beta_1}}([T_1])$ since π_1 is a continuous map. Again, by induction hypothesis, there is a $(k+1)$-covering (T_2, π_2, φ_2) of T_1 unravelling $\pi_1^{-1}(X_1)$. In this way one defines recursively $(T_{i+1}, \pi_{i+1}, \varphi_{i+1})$, which is a $(k+i)$-covering of T_i unravelling $\pi_i^{-1} \circ ... \circ \pi_1^{-1}(X_i)$.
Now, let T_∞ and the maps $\pi_{\infty,i}$ and $\varphi_{\infty,i}$ be as in Proposition 2.11. It follows that $(T_\infty, \pi_{\infty,0}, \varphi_{\infty,0})$ unravels each X_i, since :

$$\pi_{\infty,0}^{-1}(X_i) = \pi_{\infty,i+1}^{-1} \circ \pi_{i+1}^{-1} \circ ... \circ \pi_1^{-1}(X_i)$$

and, by construction, $\pi_{i+1}^{-1} \circ ... \circ \pi_1^{-1}(X_i)$ is clopen. Thus, $\pi_{\infty,0}^{-1}(X) = \bigcup_i \pi_{\infty,0}^{-1}(X_i)$ is an open set in $[T_\infty]$. We can now use Proposition 2.10 and consider a k-covering $(\tilde{T}, \pi, \varphi)$ of T_∞ which unravels $\pi_{\infty,0}^{-1}(X)$. It follows that $(\tilde{T}, \pi_{\infty,0} \circ \pi, \varphi_{\infty,0} \circ \varphi)$ is a k-covering of T that unravels X, as we wanted to prove. ∎

It remains to prove both Proposition 2.10 and Proposition 2.11 :

Proof. (*of Proposition 2.10*) : We start by recalling some notation which will be used throughout the proof : Given a tree T on a set A and some finite sequence $u \in A^{<\omega}$, $T_u = \{s \in A^{<\omega} : u \frown s \in T\}$. For a closed subset $X \subseteq [T] \subseteq A^\omega$ we let $X_u = \{x : u \frown x \in X\}$, hence $X_u \subseteq [T_u]$. Moreover, we will denote by T_X the tree defined by X, i.e $s \in T_X$ if and only if there is some $x \in X$ such that $s \subseteq x$, hence $T_X \subseteq T$.

Step 1 : In order to define $\tilde{T}$, we shall describe its legal positions. The tree $\tilde{T}$ is defined on a certain set $\tilde{A}$ which we will formally define after describing its legal positions.

In the games on $\tilde{T}$, both players play accordingly with T in the first $(k-1)$ turns. Thus, P1 and P2 choose moves x_i such that $(x_0, ..., x_i) \in T$ for $i \leq 2k-1$. Then, P1 plays $(x_{2k}, \mathcal{Q}_1)$ such that $(x_0, ..., x_{2k}) \in T$ and $\mathcal{Q}_1$ is a quasistrategy for P1 in $T_{(x_0,...,x_{2k})}$ (under the convention that P2 starts playing first on $T_{(x_0,...,x_{2k})}$). In its next move, P2 has two options :

(1) P2 plays (x_{2k+1}, u) such that $(x_0, ..., x_{2k+1}) \in T$ and u is a sequence of even length such that $u \in T_{(x_0,...,x_{2k+1})}$ and $u \in (\mathcal{Q}_1)_{(x_{2k+1})} \setminus (T_X)_{(x_0,...,x_{2k+1})}$. If $(\mathcal{Q}_1)_{(x_{2k+1})} = (T_X)_{(x_0,...,x_{2k+1})}$, we let u be the empty sequence. If P2 decides to play like this, then P1 and P2 play $x_{2k+2}, x_{2k+3}, ...$ in a way such that $(x_0, ..., x_j) \in T$ (for all j) and $u \subseteq (x_{2k+2}, x_{2k+3}, ...)$, i.e the next moves of P1 and P2 are consistent with T and extend u.

(2) P2 plays $(x_{2k+1}, \mathcal{Q}_2)$ such that $(x_0, ..., x_{2k+1}) \in T$ and $\mathcal{Q}_2$ is a quasis-

trategy for P2 in $(\mathcal{Q}_1)_{(x_{2k+1})}$ and $\mathcal{Q}_2 \subseteq (T_X)_{(x_0,...,x_{2k+1})}$. If P2 decides to play like this, then P1 and P2 play their next moves $x_{2k+1}, x_{2k+2}, ...$ such that $(x_{2k+1}, x_{2k+2}, ..., x_j) \in \mathcal{Q}_2$ (for all $j \geq 2k+2$).

Thus, the tree $\tilde{T}$ consists of all finite sequences of the form :

$$(x_0, ..., x_{2k-1}, (x_{2k}, \mathcal{Q}_1), (x_{2_{k+1}}, (1, u)), x_{2k+2}, ..., x_l)$$

or

$$(x_0, ..., x_{2k-1}, (x_{2k}, \mathcal{Q}_1), (x_{2k+1}, (2, \mathcal{Q}_2)), x_{2k+2}, ..., x_l)$$

under the restrictions previously defined. Thus, $\tilde{T}$ is a tree on a certain set $\tilde{A}$ which we define now. Let :

$\mathcal{C} = \{\mathcal{Q} : \mathcal{Q}$ is a quasistrategy for P1 in $T_{(x_0,...,x_{2k})}$ for some $(x_0, ..., x_{2k}) \in T$ under the convention that P2 plays first in $T_{(x_0,...,x_{2k})}\}$

$A_1 = \{(a, (1, u)) : a \in A,\ u \in A^{<\omega}$ of even length such that $u \in T_s$ for some $s \in T$ and $u \in (\mathcal{Q})_{(a)} \setminus (T_X)_s$, for some $\mathcal{Q} \in \mathcal{C}\}$

$A_2 = \{(a, (2, \tilde{\mathcal{Q}})) : a \in A$ and $\tilde{\mathcal{Q}}$ is a quasitrategy for P2 in $(\mathcal{Q})_{(a)}$, for some $\mathcal{Q} \in \mathcal{C}$ and such that $\tilde{\mathcal{Q}} \subseteq (T_X)_s$ for some $s \in T\}$

Then, let :

$$\tilde{A} = A \cup A_1 \cup A_2 \cup \{(a, \mathcal{Q}) : a \in A, \mathcal{Q} \in \mathcal{C}\}$$

We note that $\tilde{T}$ is a non-empty pruned tree since each player has a legal move at each turn.

Step 2 : We now define the map $\pi : \tilde{T} \to T$ and recall that we denote, with a slight abuse of notation, its extension $[\tilde{T}] \to [T]$ also by π :

$$\pi((x_0, ..., (x_{2k}, \bullet), (x_{2k+1}, \bullet), x_{2k+2}, ..., x_l)) = (x_0, ..., x_{2k}, x_{2k+1}, x_{2k+2}, ..., x_l)$$

We prove that $\pi^{-1}(X)$ is clopen in $[\tilde{T}]$:

First, we show that $\pi^{-1}(X)$ is open. Indeed, let $\tilde{x} \in \pi^{-1}(X)$ and note that $\tilde{x}(2k+1) = (x_{2k+1}, (2, \mathcal{Q}_2))$: otherwise, $\tilde{x}(2k+1) = (x_{2k+1}, (1, u))$ and $u \subseteq (x_{2k+2}, x_{2k+3}, ...)$. But since $(x_{2k+2}, x_{2k+3}, ...) \in X_{(x_0,...,x_{2k+1})}$ it would follow that $u \in (T_X)_{(x_0,...,x_{2k+1})}$, contradicting the rules of games in $\tilde{T}$. We take $\mathcal{U} = \Sigma(\tilde{x}|_{2k+1}) \cap [\tilde{T}]$ and show that $\tilde{x} \in \mathcal{U} \subseteq \pi^{-1}(X)$. Indeed, let $\tilde{y} \in \mathcal{U}$. In particular, $\tilde{y}(2k+1) = (x_{2k+1}, (2, \mathcal{Q}_2))$ and it follows that $(\tilde{y}(2k+2), \tilde{y}(2k+3), ..., \tilde{y}(l)) \in \mathcal{Q}_2$ for all $l \geq 2k+2$ according to the rules of $\tilde{T}$. Since $\mathcal{Q}_2 \subseteq (T_X)_{(x_0,...,x_{2k+1})}$, we

conclude that $\tilde{y} \in \pi^{-1}(X)$.
Furthermore, $\pi^{-1}(X)$ is closed : let $\tilde{x} \in [\tilde{T}] \setminus \pi^{-1}(X)$, so that $x := \pi(\tilde{x}) = (x_0, x_1, ...) \in [T] \setminus X$. Since X is closed, there is some N such that $\Sigma(x|_N) \subseteq [T] \setminus X$ and it follows that $[\tilde{T}] \setminus \pi^{-1}(X)$ is open.

Step 3 : Now, given a strategy $\tilde{\sigma}$ on $\tilde{T}$, we define a strategy $\sigma = \varphi(\tilde{\sigma})$ on T in a way such that for any $x \in [\sigma]$ there is some $\tilde{x} \in [\tilde{\sigma}]$ such that $\pi(\tilde{x}) = x$. It will be clear from our description that $\sigma|_n$ depends only on $\tilde{\sigma}|_n$.

Case 1 *- $\tilde{\sigma}$ is a strategy for P1 on $\tilde{T}$* : For the first $2k$ moves, σ just copies $\tilde{\sigma}$. Next, $\tilde{\sigma}$ makes P1 play $(x_{2k}, \mathcal{Q}_1)$ on $\tilde{T}$ and we let σ request P1 to play x_{2k} on T. Then, P2 replies to P1 on T with some x_{2k+1}. It follows by Theorem 2.3 that there are two possibilities :

Subcase 1.1 : P1 has a winning strategy in $G((\mathcal{Q}_1)_{(x_{2k+1})}, [(\mathcal{Q}_1)_{(x_{2k+1})}] \setminus X_{(x_0,...,x_{2k+1})})$.

Subcase 1.2 : P2 has a winning strategy in $G((\mathcal{Q}_1)_{(x_{2k+1})}, [(\mathcal{Q}_1)_{(x_{2k+1})}] \setminus X_{(x_0,...,x_{2k+1})})$.

Note that if $(\mathcal{Q}_1)_{(x_{2k+1})} = (T_X)_{(x_0,...,x_{2k+1})}$, then we are automatically in the Subcase 1.2, since $[(\mathcal{Q}_1)_{(x_{2k+1})}] = X_{(x_0,...,x_{2k+1})}$. Moreover, in this case every position is not losing for P2. We will now analyze both subcases independently :

Subcase 1.1 : We let σ follow P1's winning strategy and thus, after finitely many moves a shortest position u of even length is reached such that $u \notin (T_X)_{(x_0,...,x_{2k+1})}$. Let $u = (x_{2k+2}, ..., x_{2l-1})$. Note that :

$$(x_0, ..., (x_{2k}, \mathcal{Q}_1), (x_{2k+1}, (1, u)), x_{2k+2}, ..., x_{2l-1})$$

is a legal position of $\tilde{T}$. Hence, from then on we require σ to copy $\tilde{\sigma}$ once again.

Subcase 1.2 : We fix $\mathcal{Q}_2$ to be the canonical quasistrategy for P2 in this game. It is worth to note once again that if $(\mathcal{Q}_1)_{(x_{2k+1})} = (T_X)_{(x_0,...,x_{2k+1})}$, then any position is not losing for P2. We will assume that P2 played $(x_{2k+1}, (2, \mathcal{Q}_2))$ on $\tilde{T}$ and we define σ to copy $\tilde{\sigma}$. Note that this is possible as long as P2 plays in T in a way such that $(x_{2k+2}, ..., x_{2l-1}) \in (\mathcal{Q}_2)_{(x_0,...,x_{2k+1})}$ in order to define a legal position on $\tilde{T}$. If, at any point, P2 plays in T in such a way that $(x_{2k+2}, ..., x_{2l-1}) \notin (\mathcal{Q}_2)_{(x_0,...,x_{2k+1})}$, then it follows by definition of canonical quasistrategy that P1 has a winning strategy in $G((\mathcal{Q}_1)_{(x_{2k+1},...,x_{2l-1})}, [(\mathcal{Q}_1)_{(x_{2k+1},...,x_{2l-1})}] \setminus X_{(x_0,...,x_{2l-1})})$. If this happens, then P1 can continue by σ as in Subcase 1.1.

Case 2 *: $\tilde{\sigma}$ is a strategy for P2 on $\tilde{T}$* : Again, we request σ to follow $\tilde{\sigma}$ for the first $2k$ moves. Then, P1 plays x_{2k} on T. We define the following sets :

$$S = \{\mathcal{Q}_1 \text{ is a quasistrategy for P1 in } T_{(x_0,...,x_{2k})}\}$$

$U = \{(x_{2k+1}) \frown u \in T_{(x_0,...,x_{2k})}$: u has even length and there is some $\mathcal{Q} \in S$ such that $\tilde{\sigma}$ requires P2 to play $(x_{2k+1}, (1, u))$ when P1 plays $(x_{2k}, \mathcal{Q}_1)\}$

$$\mathcal{U} = \{x \in [T_{(x_0,...,x_{2k})}] : \text{there is some } (x_{2k+1}) \frown u \in U : (x_{2k+1}) \frown u \subseteq x)\}$$

We note that $\mathcal{U}$ is an open set of $[T_{(x_0,...,x_{2k})}]$ and we consider the game on $T_{(x_0,...,x_{2k})}$ where P2 plays first and wins if and only if $(x_{2k+1}, x_{2k+2}, ...) \in \mathcal{U}$. Let G denote this game. By Theorem 2.3, there are two possibilities to consider :

Subcase 2.1 : P2 has a winning strategy on G.

Subcase 2.2 : P1 has a winning strategy on G.

We consider each case independently :

Subcase 2.1 : After x_{2k}, σ follows this winning strategy in T until $(x_{2k+1}, ..., x_{2l-1}) \in U$. Let $u = (x_{2k+2}, ..., x_{2l-1})$ and $\mathcal{Q}_1$ witness that $u \in U$. From then on, i.e for $(x_{2l}, ...)$, P2 can play σ just by following $\tilde{\sigma}$ on :

$$(x_0, ..., x_{2k-1}, (x_{2k}, \mathcal{Q}_1), (x_{2k+1}, (1, u)), x_{2k+2}, ..., x_{2l-1})$$

Subcase 2.2 : We fix $\mathcal{Q}_1$ to be P1's canonical quasistrategy on this game. We note that if P1 played $(x_{2k}, \mathcal{Q}_1)$ on $\tilde{T}$, then $\tilde{\sigma}$ does not request P2 to play something of the form $(x_{2k+1}, (1, u))$, otherwise $(x_{2k+1}) \frown u \in U$ and by the rules of $\tilde{T}$, $(x_{2k+1}) \frown u \in \mathcal{Q}_1$ which contradicts that no sequence in $\mathcal{Q}_1$ can be in U, due to the fact that $\mathcal{Q}_1$ is P1's canonical quasistrategy. Hence, if P1 played $(x_{2k}, \mathcal{Q}_1)$, $\tilde{\sigma}$ requests P2 to play $(x_{2k+1}, (2, \mathcal{Q}_2))$ on $\tilde{T}$ and we let σ to make P2 play x_{2k+1} on T. Then, as long as P1 plays in T such that $(x_{2k+2}, ..., x_{2l}) \in \mathcal{Q}_2$, σ just follows $\tilde{\sigma}$. However, if for some l P1 plays in T such that $(x_{2k+2}, ..., x_{2l}) \notin \mathcal{Q}_2$, it follows that $(x_{2k+2}, ..., x_{2l}) \notin (\mathcal{Q}_1)_{(x_{2k+1})}$: indeed, since $\mathcal{Q}_2$ is a quasistrategy for P2 in $(\mathcal{Q}_1)_{(x_{2k+1})}$, the moves of P1 are unrestricted as long as they remain in $\mathcal{Q}_1$. Hence, in this case we are back to the Subcase 2.1. ■

Finally, we sketch the construction of the inverse limit of Proposition 2.11, ommiting all details that follow from the definitions :

Proof. (of Proposition 2.11) : We define $s \in T_\infty$ if and only if $s \in T_i$ for any i such that $|s| \leq 2(k+i)$. Moreover, if $|s| \leq 2(k+i)$ we set $\pi_{\infty,i}(s) = s$. On the other hand, if $|s| > 2(k+i)$ and j is such that $|s| \leq 2(k+j)$, we set $\pi_{\infty,i}(s) = \pi_{i+1} \circ ... \circ \pi_j(s)$. It is easy to verify that this definition is independent of our choice of j.
Finally, let σ_∞ be a strategy for T_∞. We let $\varphi_{\infty,i}(\sigma_\infty)|_{2(k+i)} = \sigma_\infty|_{2(k+i)}$ and for $j > i$ we define $\varphi_{\infty,i}(\sigma_\infty)|_{2(k+j)} = \varphi_{i+1} \circ ... \circ \varphi_j(\sigma_\infty|_{2(k+j)})$. It is then routine work to check that all conditions for the triple $(\tilde{T}_\infty, \pi_{\infty,i}, \varphi_{\infty,i})$ to be a $(k+i)$-covering of T_i hold. ■

2.3 Lipschitz and Wadge hierarchies

2.3.1 Games, order properties and Γ-hardness

We introduce the concept of Lipschitz and Wadge degrees and we prove that the set of these degrees is well-ordered in a natural way. Moreover, we introduce the idea of Γ-hardness (and Γ-completeness) which is of fundamental importance in the third section of this thesis, as a often used tool in order to identify the complexity (in Borel and projective hierarchies) of sets of interest.

Notation 2.12. Given any pointclass Γ, Γ-Det is the following statement :

For any $A \in \Gamma(\omega^\omega)$, the game $G(\omega, A)$ is determined

Definition 2.13. A function $f : \omega^\omega \to \omega^\omega$ is said to be Lipschitz if whenever $f(x) = y$, then for any $n \in \omega$ one has that $\Sigma(x|_n) \subseteq f^{-1}(\Sigma(y|_n))$.

Definition 2.14. Let $A, B \subseteq \omega^\omega$. One say that A is Wadge (Lipschitz) reducible to B if there is some continuous (Lipschitz) $f : \omega^\omega \to \omega^\omega$ such that $f^{-1}(B) = A$ and we denote it by $A \leq_W B$ ($A \leq_L B$).

Definition 2.15. Given $A, B \subseteq \omega^\omega$, we say that A and B are Wadge (Lipschitz) equivalent if $A \leq_W B$ and $B \leq_W A$ ($A \leq_L B$ and $B \leq_L A$) and we denote it by $A \equiv_W B$ ($A \equiv_L B$).

One can easily check that $\equiv_W$ and $\equiv_L$ are equivalence relations on $\mathcal{P}(\omega^\omega)$ and their elements are called Wadge and Lipschitz **degrees** respectively. Given $A \subseteq \omega^\omega$ we denote its Wadge and Lipschitz degree by $[A]_W$ and $[A]_L$ respectively.

Remark 2.16. Note that we can extend the notion of Wadge reducibility for any topological space. Indeed, we can consider any topological spaces X and Y and subsets $A \subseteq X$ and $B \subseteq Y$ and define A to be Wadge reducible to B if there is some continuous map $f : X \to Y$ such that $A = f^{-1}(B)$. The reason why we focus on ω^ω is two fold :
On one hand, we can provide a much more detailed characterization of these equivalence classes and in particular, we will prove that the ordering induced by $\leq_W$ on the set of Wadge equivalence classes is a well (semi)ordering. This is false even for Polish spaces like $\mathbb{R}$. Furthermore, we will also prove that there are no antichains of size greater than 2, with respect to the ordering $\leq_W$ of the set of Wadge equivalence classes of ω^ω. This contrasts with the fact that for any metric space (X, d) of positive topological dimensional, there are uncountably many Borel subsets of X which are pairwise incomparable with respect to $\leq_W$ (see Theorem 2.30).
On the other hand, the study of Wadge equivalence classes on $\mathcal{P}(\omega^\omega)$ is enough to understand Wadge reducibility on all zero-dimensional spaces. More concretely, if X is a zero-dimensional space and $A \subseteq X$, then one can show that there is some $B \subseteq \omega^\omega$ such that $A \equiv_W B$: Indeed, suppose that $\varphi : X \to [T]$ is a homeomorphism for some pruned tree T on ω, which exists by Theorem 4.68 and let $\iota : [T] \hookrightarrow \omega^\omega$ be the inclusion map. Moreover, let $f : \omega^\omega \to [T]$

be a continuous map which restriction to $[T]$ is the identity, which exists by Theorem 4.71. We let $B = \varphi(A)$ and note that $B \leq_W A$ by using $\varphi^{-1} \circ f$ and that $A \leq_W B$ by using $\iota \circ \varphi$. Hence, in order to study the Wadge equivalence classes on any zero dimensional Polish space, it is enough to understand the picture in ω^ω.

Intuitively, if A is reducible to B that means that A is of at most the same *complexity* as B. Indeed, if $B \in \Gamma$ for any pointclass closed under continuous maps, such as any $\mathbf{\Sigma^0_\alpha}$ or $\mathbf{\Pi^0_\alpha}$, and if A is reducible to B, then $A \in \Gamma$. To the set of Wadge (Lipschitz) degrees with the partial ordering given by $\leq_W$ ($\leq_L$) we call the Wadge (Lipschitz) hierarchy. The main goal of this section is to study some properties of these hierarchies and in order to do so we will introduce the Wadge and the Lipschitz games.

Remark 2.17. Note that Lipschitz reducibility implies Wadge reducibility, but not conversely. Indeed, let $x, y \in \omega^\omega$ given by $x = (0, 0, 0, ...)$ and $y = (0, 1, 0, ...)$. Take $A = \Sigma(x|_1)$ and $B = \Sigma(x|_2)$. Clearly, $B \leq_W A$ by the continuous reduction:

$$x \mapsto (x(0) + x(1), x(2), x(3), ...)$$

However, $B \nleq_L A$ since any Lipschitz map f with $f^{-1}(A) = B$ would be such that $f(\Sigma(y|_2)) \subseteq A$, contradicting the definition of reduction.

Remark 2.18. Note that ω^ω and $\emptyset$ are $\leq_W$ and $\leq_L$ minimal. Furthermore, if $C \subsetneq \omega^\omega$ is a non empty clopen and $B \subsetneq \omega^\omega$ is non empty, then $C \leq_W B$. Indeed, fix any $b_0 \in B$ and $b_1 \notin B$. We can use the continuous reduction $f : \omega^\omega \to \omega^\omega$ given by :

$$f(x) = \begin{cases} b_0, & \text{if } x \in C \\ b_1, & \text{if } x \notin C \end{cases}$$

Now, we introduce the Lipschitz and the Wadge games. We look at these games as games with rules, hence the notion of strategy is well defined.

For $A, B \subseteq \omega^\omega$ we introduce the Lipschitz game $GL(A, B)$. There are two players, P1 and P2 that pick respectively some $x(i) \in \omega$ and $y(i) \in \omega$ at each turn. This will yield two sequences, say $x = (x(i))_{i\in\omega}, y = (y(i))_{i\in\omega} \in \omega^\omega$ and P2 wins if and only if $x \in A \iff y \in B$. Otherwise, P1 wins.

Using previous terminology, one can consider a payoff set which consists of sequences whose coordinates on even positions are given by x_i, on odd positions by y_i and such that they satisfy the winning criteria.

Player	Moves			
P1	$x(0)$		$x(1)$	
P2		$y(0)$		$y(1)$

We also introduce the Wadge game $GW(A, B)$. The only difference between $GL(A, B)$ and $GW(A, B)$ is that on the Wadge game, P2 can decide to not pick any natural number. In the end of the game, if there is some n such that $y(n)$ is undefined, then P2 loses. Otherwise, P2 wins if and only if $x \in A \iff y \in B$. One can easily code $GW(A, B)$ by letting 0 codify the *pass* moves of P2 and $n+1$ codify $y(i) = n$, for every $i, n \in \omega$. However, simply for the sake of simplicity, whenever we define a strategy for P2 on $GW(A, B)$ we will argue within the terminology of *pass* move.

Player	Moves						
P1	$x(0)$	$\dots$	$x(n)$		$\dots$	$x(m)$	
P2				$y(0)$			$y(1)$

Proposition 2.19. Let $A, B \subseteq \omega^\omega$. Then the following hold :

(i) $A \leq_W B$ if and only if P2 has a winning strategy on $GW(A, B)$.

(ii) $A \leq_L B$ if and only if P2 has a winning strategy on $GL(A, B)$.

Proof. (i) Let τ be a winning strategy for P2 on $GW(A, B)$ and let $f : \omega^\omega \to \omega^\omega$ such that $f(x) = y$ where y is what P2 plays following τ if P1 plays x. Since τ is winning, f is well-defined and $x \in A$ if and only if $y \in B$ so it is enough to prove that f is continuous. Indeed, for all k there is some turn of this game, say n_k, at which τ makes P2 play some $y(k)$. Thus, and since τ is a strategy, $f(\Sigma(x|_{n_k})) \subseteq \Sigma(y|_k)$.
Conversely, suppose that $A \leq_W B$ by some continuous reduction f. We define a strategy τ for P2 in the following way : P2 waits until P1 have played $(x(0), ..., x(n_0))$ such that there is some $a_0 \in \omega$ such that $f(\Sigma((x(0), ..., x(n_0)))) \subseteq \Sigma(a_0)$. Then, P2 plays $y(0) = a_0$. Clearly, since f is continuous, this is possible. After P2 plays a_0 we let P1 plays until there is some $n_1 > n_0$ and $a_1 \in \omega$ such that $f(\Sigma((x(0), ..., x(n_1)))) \subseteq \Sigma((a_0, a_1))$. Again, this is possible since f is continuous. At this point, τ requests P2 to play $y(1) = a_1$. We define τ by induction and by following this construction. Furthermore, it is clear that τ is a winning strategy for P2. Indeed, if P2 decides to pick some $y(k)$ at the turn n_k of this game, we have that $f(\Sigma((x(0), ..., x(n_k)))) \subseteq \Sigma((y(0), ..., y(k)))$ by construction. Since f is a reduction, we have that $x \in A$ if and only if $y \in B$.

(ii) Let τ be a strategy for P2 on $GL(A, B)$. We define f as in (i) and in this case, since P2 always picks some $y(k)$ after P1 picked some $x(k)$, we have that $\Sigma((x|_n)) \subseteq f^{-1}(\Sigma(y|_n))$ since τ is a strategy and thus, f is Lipschitz.
Conversely, suppose that $A \leq_L B$ by some Lipschitz reduction f. We define a strategy τ for P2 in the following way : Suppose that P1 plays $x(0)$ and let $z = x(0) \frown 0^\omega$, where 0^ω denotes a sequence of zeros. Then, we define $w = f(z)$ and τ requests P2 to play $y(0) = w(0)$. We define τ inductively in this way, i.e suppose that P1 played $s \in \omega^{<\omega}$ at the end of the nth-turn and that P2 replied, following τ, with $t \in \omega^{<\omega}$. In the next turn, P1 plays some m and we let $z = s \frown m \frown 0^\omega$, $w = f(z)$ and τ requests P2 to play $y(n+1) = w(n+1)$. Note that since f is Lipschitz, if $y = \tau(x)$, then $y = f(x)$. Otherwise, there is some

N such that $y(N) \neq f(x)(N)$. However, $y(N) = f(x|_N \frown 0^\omega) \in f(\Sigma(x|_N))$ and since f is Lipschitz, this implies that $y(N) = f(x)(N)$. Hence, and since f is a reduction, it follows that τ is a winning strategy for P2. ■

A small, yet crucial observation, is the following :

Remark 2.20. Suppose that Γ-Det holds for some pointclass $\Gamma(\omega^\omega)$ which is closed under continuous preimages, finite unions and intersections. Let $\Delta(\Gamma) = \Gamma \cap \neg\Gamma$. Then, for all $A, B \in \Delta(\Gamma)$ the game $GL(A, B)$ is determined. Indeed, suppose that π_1 and π_2 are the projections onto even and odd coordinates of elements in ω^ω respectively. Consider the following set :

$$C := \{z \in \omega^\omega : \pi_1(z) \in A \text{ if and only if } \pi_2(z) \in B\}$$

Let $C_1 = \pi_1^{-1}(A) \cap \pi_2^{-1}(B)$ and $C_2 = \pi_1^{-1}(\neg A) \cap \pi_2^{-1}(\neg B)$. Clearly, $C = C_1 \cup C_2$. Since $A, B \in \Delta(\Gamma)$ and since Γ is closed under continuous preimages (each π_i is continuous) and finite intersections and unions, we easily conclude that $C \in \Delta(\Gamma)$. Thus, assuming Γ-Det, it follows that the game $G(\omega, C)$ is determined and consequently, so is $GL(A, B)$. We note that we actually only required that sets in $\Delta(\Gamma)$ are determined.

Hence, we have the following :

Proposition 2.21. $GL(A, B)$ is always determined for Borel sets $A, B \subseteq \omega^\omega$.

Proof. The result follows immediately from Remark 2.20 and Theorem 2.4. ■

The next result, known as Wadge's Lemma, is of cornerstone importance :

Theorem 2.22. *Assume Γ-Det and that Γ is closed under continuous preimages, finite unions and intersections. Then, for all $A, B \in \Delta(\Gamma)$ either $A \leq_L B$ or $B \leq_L \omega^\omega \setminus A$. In particular, either $A \leq_W B$ or $B \leq_W \omega^\omega \setminus A$.*

Proof. By Remark 2.20, $GL(A, B)$ is determined. Hence, if P2 has a winning strategy, it follows from Proposition 2.19 that $A \leq_L B$ and thus, $A \leq_W B$. Otherwise, let σ be a winning strategy for P1 on $GL(A, B)$ and define $f : \omega^\omega \to \omega^\omega$ such that $f(y)$ is what P1 plays by following σ when P2 plays y. Since σ is a strategy, then f is Lipschitz. Furthermore, σ is winning for P1, hence $B \leq_L \omega^\omega \setminus A$ and consequently, $B \leq_W \omega^\omega \setminus A$. ■

Remark 2.23. Note that by Theorem 2.4, the Theorem 2.22 holds for Borel sets, i.e if $A, B \subseteq \omega^\omega$ are Borel sets, then either $A \leq_L B$ or $B \leq_L \omega^\omega \setminus A$ (and similarly with $\leq_W$).

Notation 2.24. For convenience, we denote the set of Wadge (Lipschitz) degrees of Borel subsets of ω^ω by $WADGE_B$ ($LIPS_B$).

Corollary 2.25. Any antichain in $(WADGE_B, \leq_W)$ or in $(LIPS_B, \leq_L)$ has size at most 2.

Proof. Suppose that $A \perp_W B$ and $B \perp_W C$, for $A, B, C \in WADGE_B$. By Theorem 2.22, $B \leq_W \omega^\omega \setminus A$ and $C \leq_W \omega^\omega \setminus B$. It follows that $\omega^\omega \setminus C \leq_W B$ (simply using any continuous reduction witnessing $C \leq_W \omega^\omega \setminus B$). By transitivity, $\omega^\omega \setminus C \leq_W \omega^\omega \setminus A$ and thus, $C \leq_W A$ (again, by using the same continuous reduction of $\omega^\omega \setminus C \leq_W \omega^\omega \setminus A$). The proof is the same for Lipschitz degrees. ■

Remark 2.26. It is known that if (X, d) is a metric topological space with positive topological dimension, then there are uncountable many pairwise incomparable Borel degrees in X. ([22])

In what follows, and for the sake of readability, we will use a certain notation for the set theoretical complement. Indeed, if X is a set and $A \subseteq X$, we will often denote $X \setminus A$ by $\neg A$ (which assumes that X is understood from the context).

In order to motivate the next definition, we note that $(WADGE_B, \leq_W)$ is partially ordered and in fact, Theorem 2.22 shows us precisely what are the incomparable sets. Indeed, let $X = 2^\omega$. It can happen that $[A]_W = [\neg A]_W$, by taking $A = \Sigma(0)$. However, it may happen that $[A]_W \neq [\neg A]_W$, even if $A \neq X$. Indeed, consider any countable and dense subset $A \subseteq X$ and let $B = \neg A$. Suppose that there is some continuous reduction f such that $f^{-1}(A) = B$. It follows that $X = \bigcup_{a \in A}\{a\} \cup \bigcup_{a \in A} f^{-1}(a)$. Note that each $f^{-1}(a)$ has empty interior : suppose towards a contradiction that $b \in \mathcal{U} \subseteq f^{-1}(a)$ for some open set $\mathcal{U}$ and $a \in A$. Since A is dense, there is some $c \in \mathcal{U} \cap A$ and thus, $c \in B \cap A$ which is impossible. Hence, X is a countable union of nowhere dense sets which contradicts the Baire Category Theorem. Thus, $B \not\leq_W A$ and since $B = \neg A$, we conclude that A and B are incomparable. Furthermore, the same argument works for Lipschitz degrees.

However, if A and B are Borel, then Theorem 2.22 implies that $[A]_W$ and $[B]_W$ are incomparable precisely when $B = \neg A$ and $[A]_W \neq [\neg A]_W$. Similarly, with Lipschitz degrees. This motivates the following definition of coarse Wadge (Lipschitz) degrees :

$$\text{For } A \subseteq \omega^\omega \text{ Borel, let } A^* = [A]_W \cup [\neg A]_W$$

We denote the set of coarse Wadge degrees by $WADGE_B^*$ and let :

$$A^* \leq_W^* B^* \text{ if and only if } A \leq_W B \text{ or } A \leq_W \neg B$$

Similarly, we define coarse Lipschitz degrees and denote this set by $LIPS_B^*$ and define analogously the ordering $\leq_L^*$.

Theorem 2.27. $(LIPS_B^*, \leq_L^*)$ *is a well ordering.*

Proof. By the previous comments, we already know that $\leq_L^*$ is a total order, hence it is enough to show that there is no infinite descending chain. For the sake of simplicity, throughout this proof, we will use $\leq^*$ instead of $\leq_B^*$. Suppose, by contradiction, that such chain exists :

$$A_0^* >^* A_1^* >^* A_2^* >^* ... > A_n^* > A_{n+1}^* > ...$$

By Theorem 2.4 the game $GL(A_n, A_{n+1})$ is determined. Hence, it follows by Proposition 2.19 that P1 has a winning strategy on $GL(A_n, A_{n+1})$ otherwise, $A_n \leq_L A_{n+1}$. Similarly, P1 has a winning strategy on $GL(A_n, \neg A_{n+1})$. We let σ_n^0 and σ_n^1 be, respectively, the winning strategies of P1 on $GL(A_n, A_{n+1})$ and on $GL(A_n, \neg A_{n+1})$. For all n, let $A_n^0 = A_n$ and $A_n^1 = \neg A_n$. Now, fix any $x \in 2^\omega$. We define the following plays on infinitely many games running simultaneously :

Player	Moves				
P1 ($\sigma_0^{x(0)}$)	y_0^0		y_1^0		y_2^0
P2		y_0^1		y_1^1	
P1 ($\sigma_1^{x(1)}$)	y_0^1		y_1^1		y_2^1
P2		y_0^2		y_1^2	
P1 ($\sigma_2^{x(2)}$)	y_0^2		y_1^2		y_2^2
P2		y_0^3		y_1^3	
...	....		...		...

In words, P1 starts the nth-game with y_0^n by following $\sigma_n^{x(n)}$. Then, P2 replies to P1 in the nth-game by playing y_0^{n+1}. Then, P1 replies by playing y_1^n while following $\sigma_n^{x(n)}$ in the nth-game. We let P2 play y_1^{n+1} as a response, an so on.

Furthermore, define $y_n(x) := (y_k^n)_{k\in\omega}$, i.e a play of P1 in the nth-game. By construction, and since $\sigma_n^{x(n)}$ is a winning strategy, we get that :

$$y_n(x) \notin A_n \Longleftrightarrow y_{n+1}(x) \in A_{n+1}^{x(n)} \ (*)$$

Let $X = \{x \in 2^\omega : y_0(x) \in A_0\}$. Assume for a moment the following claims :

Claim 1 : If $x, z \in 2^\omega$ and there is some k such that $x(n) = z(n)$ for all $n \neq k$ and $x(k) \neq z(k)$, then $x \in X$ if and only if $z \notin X$.

Claim 2 : Let Z be a topological space and suppose that $A \subseteq Z$ has the BP. Then, either A is meager or there is some non empty open set $\mathcal{U} \subseteq Z$ on which A is comeager. Furthermore, one and only one of the alternatives must hold if Z is a Baire space (in particular, if $Z = 2^\omega$).

We now derive a contradiction. On one hand, and since $x \mapsto y_n(x)$ is clearly continuous, X is Borel and thus, has the BP. On the other hand, it follows by Claim 2 that either X is meager or there is some $n \in \omega$ and $s \in 2^n$ such that X

is comeager in $\Sigma(s)$. Now, let $\varphi : \Sigma(s) \to \Sigma(s)$ be given by :

$$\varphi((x_i)) = (x_0, ..., x_{n-1}, 1 - x_n, x_{n+1}, ...)$$

Clearly, φ is a homeomorphism and, by Claim 1, $x \in X$ if and only if $\varphi(x) \notin X$. Hence, $\varphi(X \cap \Sigma(s)) = \neg X \cap \Sigma(s)$. But then, if $X \cap \Sigma(s)$ is comeager it follows that $\neg X \cap \Sigma(s)$ is also comeager and thus, $\Sigma(s)$ is meager, contradicting the Baire Category Theorem. Otherwise, X is meager and thus, so is $X \cap \Sigma(s)$ and $\neg X \cap \Sigma(s)$ which once again contradicts the Baire Category Theory. Therefore, X is a Borel set without the BP, which is impossible.

To finish the proof, it remains to prove Claim 1 and Claim 2 :

Proof (of Claim 1) : Note that by definition, $y_n(x)$ depends only on $x(k)$ for $k \geq n$. Hence, by $(*)$ it follows that :

$$y_k(x) \notin A_k \Longleftrightarrow y_{k+1}(x) \in A_{k+1}^{x(k)} \Longleftrightarrow y_{k+1}(z) \in A_{k+1}^{x(k)} \Longleftrightarrow y_{k+1}(z) \notin A_{k+1}^{z(k)}$$

Thus, $y_k(x) \notin A_k \iff y_k(z) \in A_k$. But if $x \in X$ if and only if $z \in X$, since $x(n) = z(n)$ for all $n < k$, one just repeats the previous argument by using $(*)$ and derive a contradiction.

Proof (of Claim 2) : Since A has the BP, then $A \Delta \mathcal{U} = M$, for some meager set M. But if A is non meager, then $\mathcal{U} \neq \emptyset$ and A is comeager in $\mathcal{U}$, since $\mathcal{U} \setminus A \subseteq M$.

■

Corollary 2.28. $(WADGE_B^*, \leq_W^*)$ is a well ordering.

Proof. Any infinite descending chain of Wadge coarse degrees is also an infinite descending chain of Lipschitz coarse degrees. Indeed, let $A, B \subseteq \omega^\omega$ be Borel and suppose that $A <_W^* B$. By Theorem 2.22, either $A \leq_L B$ or $B \leq_L \neg A$. If $B \leq_L \neg A$, then $B \leq_W \neg A$ and thus, $B \leq_W^* A$ which is impossible. Hence, $A \leq_L B$ and thus, $A \leq_L^* B$. However, if $A =_L^* B$, then we get that $B \leq_W^* A$ which is impossible. ■

Remark 2.29. The class of coarse Wadge degrees of $\mathbb{R}$ is not well ordered (c.f [10]).

In spirit of Remarks 2.26 and 2.29, we have the following result :

Theorem 2.30. *Let X be any Polish space. The following are equivalent :*

(i) X has topological dimension zero.

(ii) For any Borel sets A and B of X, either $A \leq_W B$ or $B \leq_W X \setminus A$.

(iii) The Wadge order on the Borel sets of X is a well-quasiorder.

(iv) There are at most two pairwise incomparable Borel subsets of X.

(v) There are at most countably many pairwise incomparable Borel subsets of X.

Proof. The reader can check [22]. ∎

In what remains of this section, we will describe the Wadge and Lipschitz hierarchies which we now know to be semi well-orderings. We will also compare these hierarchies and describe how much the Lipschitz hierarchy refines the Wadge hierarchy. But before doing so, we introduce an important concept that will be used in further sections and comment on its order properties with respect to $\leq_W$.

Definition 2.31. Let Γ be a pointclass in a Polish space and let Y be Polish. One says that $A \subseteq Y$ is Γ-hard if $B \leq_W A$ for any $B \in \Gamma(X)$ and X, a zero dimensional Polish space. If, furthermore, $A \in \Gamma(Y)$, then A is said to be Γ-complete.

Theorem 2.32. *Let X be a zero dimensional Polish space. Then :*

(i) $A \subseteq X$ is $\boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\alpha}}$-complete if and only if $A \in \boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\alpha}} \setminus \boldsymbol{\Pi}^{\mathbf{0}}_{\boldsymbol{\alpha}}$.

(ii) $A \subseteq X$ Borel is $\boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\alpha}}$-hard if and only if $A \notin \boldsymbol{\Pi}^{\mathbf{0}}_{\boldsymbol{\alpha}}$.

Proof. By Remark 2.16, we can assume that $X = \omega^\omega$. Suppose that A is $\boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\alpha}}$-hard and that $A \in \boldsymbol{\Pi}^{\mathbf{0}}_{\boldsymbol{\alpha}}$. Then, if $B \in \boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\alpha}}$, one has that $B \leq_W A$ and thus, $\boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\alpha}} \subseteq \boldsymbol{\Pi}^{\mathbf{0}}_{\boldsymbol{\alpha}}$. On the other hand, if $B \in \boldsymbol{\Pi}^{\mathbf{0}}_{\boldsymbol{\alpha}}$, then $B = \neg C$ with $C \in \boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\alpha}} \subseteq \boldsymbol{\Pi}^{\mathbf{0}}_{\boldsymbol{\alpha}}$, from where it follows that $B \in \boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\alpha}}$ and thus, $\boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\alpha}} = \boldsymbol{\Pi}^{\mathbf{0}}_{\boldsymbol{\alpha}}$, which is impossible by Theorem 1.10. Therefore, we conclude that $A \notin \boldsymbol{\Pi}^{\mathbf{0}}_{\boldsymbol{\alpha}}$. Note that since any set in a pointclass $\boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\alpha}}$ is Borel, this proves the *only if* part in both (i) and (ii).
Conversely, suppose that A is Borel such that $A \notin \boldsymbol{\Pi}^{\mathbf{0}}_{\boldsymbol{\alpha}}$ and Y is any zero dimensional Polish space with $B \in \boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\alpha}}(Y)$. By Theorem 2.22, either $A \leq_W \neg B$ or $B \leq_W A$. Thus, we have that $B \leq_W A$ and it follows that A is $\boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\alpha}}$-hard. Again, this proves the *if* part in both (i) and (ii). ∎

Remark 2.33. The statement of Theorem 2.32 remains true if we switch $\boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\alpha}}$ with $\boldsymbol{\Pi}^{\mathbf{0}}_{\boldsymbol{\alpha}}$.

Corollary 2.34. The sets in $\boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\alpha}} \setminus \boldsymbol{\Pi}^{\mathbf{0}}_{\boldsymbol{\alpha}}$ are maxima in $\leq_W$ among $\boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\alpha}}$ and the sets in $\boldsymbol{\Pi}^{\mathbf{0}}_{\boldsymbol{\alpha}} \setminus \boldsymbol{\Sigma}^{\mathbf{0}}_{\boldsymbol{\alpha}}$ are maxima in $\leq_W$ among $\boldsymbol{\Pi}^{\mathbf{0}}_{\boldsymbol{\alpha}}$.

Remark 2.35. Suppose that $A \subseteq Y$ is $\boldsymbol{\Sigma}^{\mathbf{1}}_{\mathbf{1}}$-complete. Then, by definition one has that A is analytic. Moreover, A is not Borel. Indeed, suppose towards a contradiction that A is Borel and choose any $B \subseteq \omega^\omega$ which is analytic but not Borel. Since A is $\boldsymbol{\Sigma}^{\mathbf{1}}_{\mathbf{1}}$-hard, there is a continuous reduction $B \leq_W A$ which would imply that B is also Borel. Hence, a possible strategy to show that a given set A is analytic but not Borel, is to show that A is $\boldsymbol{\Sigma}^{\mathbf{1}}_{\mathbf{1}}$-complete. Clearly, all the previous remark also applies to coanalytic sets.

2.3.2 Shape of Lipschitz and Wadge hierarchies

According to Theorem 2.27 and Corollary 2.28, one can associate an ordinal to each level of the Lipschitz and Wadge (coarse) hierarchy. We now explore the *shape* of these hierarchies for the Borel sets of ω^ω.

Definition 2.36. Let $A \subseteq \omega^\omega$. We say that A is self-dual (or its Wadge (Lipschitz) degree) if $[A]_W = [\neg A]_W$ ($[A]_L = [\neg A]_L$).

Definition 2.37. Suppose that $A \subseteq \omega^\omega$ and $\{A_i\}_{i\in\omega} \subseteq \omega^\omega$, such that $A \neq \emptyset$ and $A_i \neq \emptyset$. Then :

(i) $n \frown A_i := \{n \frown s : s \in A_i\}$

(ii) $A_i \oplus A_j := \bigcup_{n\in\omega}((2n) \frown A_i) \cup ((2n+1) \frown A_j)$

(iii) $\bigoplus_{i\in\omega} A_i := \bigcup_{i\in\omega} i \frown A_i$

(iv) $A_{[n]} := \{s \in \omega^\omega : n \frown s \in A\}$

Remark 2.38. Note that $\neg(A \oplus B) = \neg A \oplus \neg B$ and that if $A = \bigoplus_{i\in\omega} A_i$, then $\neg A = \bigoplus_{i\in\omega} \neg A_i$. Furthermore, $\neg(0 \frown A) = \bigcup_{n\geq 1}(n \frown \omega^\omega) \cup (0 \frown \neg A)$. Any of these equalities follow easily from the definitions. We just state them here for further reference.

Note that $[\emptyset]_L$ and $[\omega^\omega]_L$ are $\leq_L$ (and $\leq_W$) minimal, hence in both hierarchies, the first level (corresponding to level $\lambda = 0$) is a non self-dual degree. We will now describe what happens at the level λ of the coarse Lipschitz hierarchy of Borel subsets of ω^ω for when λ is a successor ordinal.

Proposition 2.39. If the level λ of the coarse Lipschitz hierarchy of Borel sets of ω^ω is $[A]_L$ with A non self-dual, then the level $\lambda+1$ is represented by a self dual degree.

Proof. Consider $[A \oplus \neg A]_L$. Then, we have that :

(i) $B := A \oplus \neg A$ is self dual : P2 has a winning strategy on $GL(B, \neg B)$, say τ. Indeed, if P1 starts the game by playing $k \in \omega$, then τ requests P2 to play $k+1$ and from now on, P2 just copies P1 moves. This is easily seen to be a winning strategy, using Remark 2.38. Thus, by Proposition 2.19, $B \leq_L \neg B$ and thus, $\neg B \leq_L B$ so that $B =_L \neg B$.

(ii) $A <_L A \oplus \neg A$ and $\neg A <_L A \oplus \neg A$: Note that P2 has a winning strategy τ on $GL(A, B)$. Indeed, no matter what P1 plays at the beginning, τ requests P2 to play an even natural number, say 0. Then, P2 just copies P1 moves from now on. Hence, by Proposition 2.19, we get that $A \leq_L B$. On the other hand, P1 has a winning strategy σ on $GL(B, A)$. Indeed, let σ request P1 to play 1 in the first move and then it simply copies P2 moves. Thus, by Proposition 2.19, one has that $B \not\leq_L A$. Similarly, we prove that $\neg A <_L B$.

(iii) There is no C such that $A <_L C <_L A \oplus \neg A$: Suppose that such C exists. Since $GL(B, C)$ is determined, one of the players have a winning strategy. If P2 has a winning strategy, then $B \leq_L C$. Thus, P1 has a winning strategy σ on $GL(B, C)$. There are two possible cases : either σ requests P1 to start this game by playing an even number, or by playing an odd number. Suppose that σ requests P1 to start the game with an even number. Note that if we consider the game $GL(B, C)$ from this point, i.e after P1 made its first move by playing an even number and by letting P2 *start* first, we actually are playing $GL(C, \neg A)$. But now σ will be a winning strategy on $GL(C, \neg A)$ and since the roles of P1 and P2 are switched, we have that $C \leq_L \neg A$. Thus, it follows that $A \leq_L \neg A$ which is impossible since we assumed that A is non self-dual. On the other hand, if σ requests P1 to start the game with an odd number, then P2 has a winning strategy on $GL(C, A)$ and thus, $C \leq_L A$, which is also impossible. ■

Proposition 2.40. If the level λ of the coarse Lipschitz hierarchy of Borel sets of ω^ω is $[A]_L$ with A self-dual, then the level $\lambda + 1$ is represented by a self-dual degree.

Proof. Consider $[0 \frown A]_L$. Then, we have that :

(i) $0 \frown A$ is self-dual : P2 has a winning strategy τ on $GL(0 \frown A, \neg(0 \frown A))$. Indeed, suppose that P1 starts the game with 0. Then, τ requests P2 to reply with 0 and now we are playing $GL(A, \neg A)$. But since A is self-dual, one has that $A \leq_L \neg A$ and thus, P2 has a winning strategy τ' on $GL(A, \neg A)$, so it is enough for τ to copy τ' from now on. On the other hand, suppose that P1 started the game with an integer different than zero. Then, τ request P2 to reply with 0 and with any arbitrary element of A from now on. Notice that this is possible, since $A \neq \emptyset$, as it is self-dual.

(ii) $A <_L 0 \frown A$: P2 has a winning strategy τ on $GL(A, 0 \frown A)$. Indeed, it suffices that τ requests P2 to reply to the first P1's move with a 0 and from then on, τ requests P2 to simply copy P1's moves. Furthermore, P1 has a winning strategy σ on $GL(0 \frown A, A)$ and thus, $0 \frown A \nleq_L A$. Indeed, let σ request P1 to start the game with 0 and notice that if we switch the roles of P1 and P2, we are now playing $GL(A, \neg A)$. But since A is self-dual, now σ just copies the existing winning strategy on $GL(A, \neg A)$.

(iii) There is no C such that $A <_L C <_L 0 \frown A$: Suppose that such C exists. Then, by Borel determinacy, it follows that P1 has a winning strategy σ on $GL(0 \frown A, C)$. There are two possible cases : on one hand, suppose that σ requests P1 to start by playing with 0. Then, P2 has a winning strategy on $GL(C, \neg A)$ by copying σ from the second move on, which implies that $C \leq_L \neg A$. But since A is self-dual, we get that $C \leq_L A$ which is impossible. On the other hand, suppose that σ requests P1 to start by playing with an integer different than 0. Then, P2 has a winning strategy on $GL(\neg C, \omega^\omega)$ which is also impossible. ■

Now we analyse what happens at the level λ of the coarse Lipschitz hierarchy for Borel sets of ω^ω when λ is a limit ordinal.

Theorem 2.41. *Let λ be a limit ordinal. Then, the level λ of the coarse Lipschitz hierarchy for Borel sets of ω^ω is represented by a self-dual degree if and only if $cf(\lambda) = \omega$.*

Proof. Suppose that λ is a limit ordinal such that $cf(\lambda) = \omega$ and let $\{\alpha_i\}_{i\in\omega}$ be an increasing sequence of ordinals such that $\lim \alpha_i = \lambda$. For each $i \in \omega$, let A_i represent the level α_i of the Lipschitz hierarchy and set $A = \bigoplus_{i\in\omega} A_i$. Then :

(i) A is self-dual : Indeed, P2 has a winning strategy τ on $GL(A, \neg A)$: suppose that P1 starts the game with k. Then, τ requests P2 to reply with $k+1$. Then, it is equivalent to play the game $GL(A_k, \neg A_{k+1})$. By Theorem 2.22, P2 has a winning strategy τ' on this game and thus τ just follows τ' from the second turn on. Note that we used Remark 2.38 for $\neg A$.

(ii) For all $i \in \omega$, $A_i <_L A$: Note that P2 has a winning strategy on $GL(A_i, A)$ by replying to P1's first move with i and after that, just copying P1's moves with one move delay. On the other hand, P1 has a winning strategy σ on $GL(A, A_i)$. Indeed, let σ request P1 to start by playing $i+1$. But then, it is equivalent to consider the game $GL(A_i, \neg A_{i+1})$, switching the roles of P1 and P2. Again, by Theorem 2.22, $A_i \leq_L \neg A_{i+1}$ and thus σ simply copies the winning strategy for the second player (now P1) on the game $GL(A_i, \neg A_{i+1})$, from the second turn on.

(iii) There is no C such that for all $i \in \omega$, $A_i <_L C <_L A$: Suppose that such C exists. Then, P1 has a winning strategy σ on $GL(A, C)$. If σ requests P1 to start the game with k, then it is equivalent to consider the game $GL(C, \neg A_k)$ with the roles of P1 and P2 switched. But then, P2 (now P1) has a winning strategy on $GL(C, \neg A_k)$ which implies that $C \leq_L \neg A_k$, a contradiction to our assumption.

Conversely, suppose that a self-dual set A represents the level λ of the hierarchy. We prove that $cf(\lambda) = \omega$. It is enough to define a strictly increasing sequence (with respect to $\leq_L$) of sets B_n such that $B_n <_L A$ and such that $\bigoplus_{n\in\omega} B_n = A$. Then, if each B_n represents the level λ_n, we get that $\lim_n \lambda_n = \lambda$ and thus, $cf(\lambda) = \omega$. Set $B_0 := A_{[0]}$ and then define each B_n as :

$$B_{n+1} = \begin{cases} A_{[n+1]} & \text{if} \quad B_n <_L A_{[n+1]} \\ 0 \frown B_n & \text{if} \quad B_n \not<_L A_{[n+1]} \quad \text{and} \quad B_n \quad \text{is self-dual} \\ B_n \oplus \neg B_n & \text{if} \quad \text{otherwise} \end{cases}$$

■

To summarize, the shape of the coarse Lipschitz hierarchy for Borel subsets of ω^ω is as follows :

(i) $[\emptyset]_L$ and $[\omega^\omega]_L$ are $\leq_L$-minimal and thus, the level $\lambda = 0$ is occupied by a non self-dual degree.

(ii) (Successor ordinals) Suppose that the level λ is represented by $[A]_L$. Either A is non self-dual, in which case the level $\lambda + 1$ is a self-dual degree or A is self-dual in which case the level $\lambda + 1$ is also a self-dual degree.

(iii) (Limit ordinal) Consider the level λ, with λ a limit ordinal. Then, $cf(\lambda) = \omega$ if and only if the level λ is represented by a self-dual degree.

The picture of the coarse Wadge hierarchy for Borel subsets of ω^ω is quite different. We do not provide the details, however the interested reader can consult [25] and [10]. Proposition 2.42 gives a notion of *how much the Lipschitz hierarchy refines the Wadge hierarchy.*

Given a family of degrees in the coarse Lipschitz hierarchy which is indexed by some ordinal α, say $\{[A_\beta]_L\}_{\beta<\alpha}$, we say that it is a sequence of α consecutive degrees if for every $\beta < \alpha$ the level of A_β in the hierarchy is λ, then the level of $A_{\beta+1}$ is $\lambda + 1$.

Proposition 2.42. Let $\alpha < \omega_1$. Then, every sequence of α consecutive self-dual degrees in the coarse Lipschitz hierarchy is contained in a single coarse Wadge degree.

Proof. Let $\{[A_\beta]_L\}_{\beta<\alpha}$ be a sequence of α consecutive self dual degrees. We need the following facts :

Fact 1 : Suppose that $\{A_n\}_{n\in\omega}$ is a sequence with $A_n \leq_W A$ for all n. Then, $\bigoplus_{n\in\omega} A_n \leq_W A$.

(*Proof of FACT 1*) : It is enough to prove that P2 has a winning strategy τ on $GW(\bigoplus_{n\in\omega} A_n, A)$. If P1 starts the game by playing $i \in \omega$, then τ requests P2 to skip a move. From now on, we just let τ follow the winning strategy on $GW(A_i, A)$, which exists since $A_i \leq_W A$.

Fact 2 : If $[A]_W$ is a self dual degree, then $[A]_W =_W [0 \frown A]_W$.

(*Proof of FACT 2*) : On one hand P2 has a winning strategy τ on $GW(A, 0 \frown A)$, by simply replying to P1's first move with 0 and then copying every P1 move with one turn delay. On the other hand, P2 also has a winning strategy τ on $GW(0 \frown A, A)$: if P1 starts the game by playing 0, then τ requests P2 to skip a move and then to copy each P1 moves. Otherwise, if P1 starts the game by playing something different than 0, τ requests P2 to simply play any sequence which is not in A. Note that this is possible since A is self-dual and thus, $\neg A \neq \emptyset$.

If $\alpha = 1$, we are done. Now, suppose that, by induction hypothesis, whenever $\beta < \alpha$ then $A_\beta =_W A$ (for some A) and note that since $\alpha < \omega_1$, then either $cf(\alpha) = 1$ or $cf(\alpha) = \omega$. Thus, we have two cases to consider : if α is a successor ordinal, then $[A_\alpha]_L = [0 \frown A_{\alpha-1}]_L$ by Proposition 2.40. It follows that $A_\alpha =_W A$ by Fact 2 (and induction hypothesis). Otherwise, if α is a limit ordinal, it follows by the proof of Theorem 2.41 that $A_\alpha =_L \bigoplus_{n\in\omega} A_n$ with $A_n <_L A_\alpha$ for every n. Thus, by Fact 1 and the induction hypothesis, it follows that $A_\alpha =_W A$. ■

Furthermore, we have the following result which, together with Proposition 2.42, allow us a *clear* picture of the coarse Wadge hierarchy for Borel sets of ω^ω.

Theorem 2.43. *Let λ be a level of the coarse Wadge hierarchy of Borel sets of ω^ω. Then :*

(i) If the level λ is a non self-dual degree, then the level $\lambda + 1$ is a self-dual degree.

(ii) If $cf(\lambda) = \omega$, then the level λ is a self-dual degree.

(iii) A self-dual set in the Wadge hierarchy is also self-dual in the Lipschitz hierarchy.

Proof. The interested reader can read [10] and [25]. ■

Recall that the Borel hierarchy provides a stratification of the Borel sets of a Polish space into at most ω_1 levels. If the Polish space is uncountable, then the Borel hierarchy consists precisely of ω_1 levels. The Wadge and Lipschitz hierarchies (on ω^ω), on the other hand, provide a stratification with strictly more than ω_1 levels.

Remark 2.44. One can define the following ordinal :

$$\Theta := \sup\{\alpha \in \text{Ord such that there is a surjective map } f : \omega^\omega \to \alpha\}$$

Clearly, $\Theta > \omega_1$. We show that the length of the Wadge hierarchy is at most Θ : First, note that by Theorem 4.75 one can index all continuous maps from ω^ω to ω^ω by $\{f_x\}_{x\in\omega^\omega}$. Now for a Borel set $A \subseteq \omega^\omega$, let $\lambda(A)$ be its associated level in the coarse Wadge hierarchy. Suppose that $\lambda(A) = \alpha$ and let $B \subseteq \omega^\omega$ be a Borel set such that $\lambda(B) = \beta \leq \alpha$. Hence, there is some $x_0 \in \omega^\omega$ such that $B = f_{x_0}^{-1}(A)$. It follows that $f : \omega^\omega \to \alpha + 1$ defined by $x \mapsto \lambda(f_x^{-1}(A))$ is surjective. Thus, $\sup\{\lambda(A)\} \leq \Theta$, so the length of the Wadge hierarchy is at most Θ (recall that for uncountable Polish spaces the length of the Borel hierarchy is ω_1). In fact, one can do better :

Theorem 2.45. *The length of the Wadge hierarchy is Θ.*

Proof. The interested reader can consult [26]. ■

Corollary 2.46. The length of the Lipschitz hierarchy is Θ.

Proof. By Proposition 2.42, each level of the Wadge hierarchy is contained in at most ω_1 levels of the Lipschitz hierarchy. Since by Theorem 2.45 the length of the Wadge hiearchy is Θ and since $\Theta.\omega_1 = \Theta$ (as $\Theta > \omega_1$), it follows that the length of the Lipschitz hierarchy is also Θ. ■

2.4 Games and regularity properties

2.4.1 $*$- games and $\boldsymbol{\Sigma^1_1}$-Det

We mentioned previously that the question of whether or not the Perfect Set Property (PSP) holds for all coanalytic sets of a Polish space is actually independent from ZFC. On the other hand, we proved in the section 2.2 that all Borel subsets are determined and we commented that this result is the *best* we can hope for within ZFC. In what follows, we introduce the *-game and we prove that if a subset is determined for this game, then it has the PSP. It is in this context that we motivate the axiom of $\boldsymbol{\Sigma^1_1}$-determinacy.

We begin by introducing the *-game, $G^*_2(X)$: Let $X \subseteq 2^\omega$ and at each turn, P1 picks some $s_i \in 2^{<\omega}$ and P2 picks some $j_i \in \{0, 1\}$. This will produce some element $x = s_0 \frown j_0 \frown ...$ and we say that P1 wins this game if and only if $x \in X$. Otherwise, P2 wins.

Player	Moves			
P1	s_0		s_1	
P2		j_0		j_1

Given two games, G_1 and G_2, we will say that G_1 can be *reduced* to G_2 if a winning strategy for some player in G_2 induces a winning strategy for the same player in G_1. We explain how can we reduce a game $G^*_2(X)$ to a game $G(\omega, X^*)$ (for a certain payoff set X^*).
For a moment, suppose that at each turn P1 picks elements $s_i \in \omega^{<\omega}$ and P2 picks some $j_i \in \omega$. Let's denote this game by $G^*_\omega(\tilde{X})$, for a fixed payoff set $\tilde{X} \subseteq \omega^\omega$. First, note that it is easy to reduce any game $G^*_\omega(\tilde{X})$ to a game $G(\omega, X^*)$: Indeed, suppose that $\{t_i\}_{i\in\omega}$ is an enumeration of $\omega^{<\omega}$ and consider a target set $\tilde{X} \subset \omega^\omega$. We let :

$$x \in X^* \text{ if and only if } t_{x(0)} \frown x(1) \frown t_{x(2)} \frown x(3) \frown ... \in \tilde{X}$$

Let σ^* be a winning strategy for P2 in $G(\omega, X^*)$. Then, this will induce a winning strategy $\tilde{\sigma}$ for P2 in the game $G^*_\omega(\tilde{X})$:

Suppose P1 plays $s_0 \in \omega^{<\omega}$ in its first move in $G^*_\omega(\tilde{X})$. Let $s_0 = t_{x(0)}$ and assume that P1 plays $x(0)$ in its first move in $G(\omega, X^*)$. Then, σ^* makes P2 play some $x(1) \in \omega$ and we let $\tilde{\sigma}$ request P2 to play $x(1) = j_0$ in $G^*_\omega(\tilde{X})$. Next, P1 replies with some $s_1 \in \omega^{<\omega}$ and once again, we assume that P1 played $x(2) \in \omega$ such that $s_1 = t_{x(2)}$ in $G(\omega, X^*)$. Again, σ^* will make P2 play some

$x(3) \in \omega$ in response and we define $\tilde{\sigma}$ as to request P2 to play $x(3) = j_1$ in $G^*_\omega(\tilde{X})$ in response to (s_0, j_0, s_1), and so on. It is immediate, by definition of X^*, that $\tilde{\sigma}$ is a winning strategy for P2 in $G^*_\omega(\tilde{X})$ and it is entirely analogous with winning strategies for P1. Hence, winning strategies (for a player) in games $G(\omega, X^*)$ translate to winning strategies (for the same player) in games $G^*_\omega(\tilde{X})$.

Moreover, note that the map $\Psi : \omega^\omega \to \omega^\omega$ such that $\Psi(x) = t_{x(0)} \frown x(1) \frown ...$, is continuous. Hence, if Γ-det holds for some pointclass Γ which is closed under continuous preimages, it follows that sets from that pointclass are also determined in $G^*_\omega(\tilde{X})$.

Now, in order to reduce games $G^*_2(X)$ with games $G^*_\omega(\tilde{X})$, we consider the continuous map $g : \omega^\omega \to 2^\omega$:

$$g(\alpha)(n) = \begin{cases} 0, & \text{if } \alpha(n) = 0 \\ 1, & \text{if } \alpha(n) > 0 \end{cases}$$

Moreover, given a target set $X \subseteq 2^\omega$, we let :

$$x \in \tilde{X} \text{ if and only if } g(x) \in X$$

Furthermore, suppose that $\tilde{\sigma}$ is a winning strategy for P1 in $G^*_\omega(\tilde{X})$. This again will induce a winning strategy σ for P1 in the game $G^*_2(X)$:

We will consider, by slight abuse of notation, g to be defined in the natural way to elements in $\omega^{<\omega}$. Suppose that $\tilde{\sigma}$ makes P1 start the game by playing $\tilde{s_0} \in \omega^{<\omega}$. We define σ such that it requests P1 to start the game by playing $s_0 = g(\tilde{s_0}) \in 2^{<\omega}$. Then, P2 replies by playing j_0 and we assume that P2 plays $\tilde{j_0} = j_0$ in $G^*_\omega(\tilde{X})$, so that $\tilde{\sigma}$ requests P1 to reply with some $\tilde{s_1} \in \omega^\omega$. Again, we define σ to make P1 reply to (s_0, j_0) with $s_1 = g(\tilde{s_1})$ in $G^*_2(X)$ and so on.
It is clear that σ is a winning strategy for P1 on $G^*_2(X)$ and we can proceed similarly with (winning) strategies for P2.

To summarize, suppose that Γ-det holds for some pointclass Γ which is closed under continuous preimages and consider the game $G^*_2(X)$ for $X \in \Gamma$. Since $\tilde{X} = g^{-1}(X) \in \Gamma$ and $X^* = \Psi^{-1}(\tilde{X}) \in \Gamma$, it follows that $G(\omega, X^*)$ is determined. Thus, so is the game $G^*_\omega(\tilde{X})$ and consequently, also the game $G^*_2(X)$.

Remark 2.47. It is worth to note that we can define *-games in yet, another equivalent formulation. Indeed, let X be a Polish space and thus, in particular, let $\{\mathcal{U}_n\}$ be a countable basis. It is not hard, using once again an argument with enumerations (thus, we assumed second countability) that the following game is equivalent to $G^*_2(X')$ for some X' :

In this game, we fix some subset $A \subseteq X$ and at each turn, P1 chooses a pair of open basic sets $(\mathcal{U}^{(n)}_0, \mathcal{U}^{(n)}_1)$ and P2 selects one of them, by choosing some $i_j \in 2$. Furthermore, diam$(\mathcal{U}^{(n)}_i) < 2^{-n}$, $\overline{\mathcal{U}^{(n)}_0} \cap \overline{\mathcal{U}^{(n)}_1} = \emptyset$ and $\overline{\mathcal{U}^{(n+1)}_0 \cup \mathcal{U}^{(n+1)}_1} \subseteq \mathcal{U}^{(n)}_{i_n}$.

Player	Moves			
P1	$(\mathcal{U}_0^{(0)}, \mathcal{U}_1^{(0)})$		$(\mathcal{U}_0^{(1)}, \mathcal{U}_1^{(1)})$	
P2		i_0		i_1

Since X is complete, let $x := \bigcap_n \overline{\mathcal{U}_{i_n}^{(n)}}$ and P1 wins if and only if $x \in A$. Otherwise, P2 wins. Note that with each decision by P2, the game is defining a Cantor scheme, which antecipates the connection between this game and the PSP.

For the sake of brevity, let's denote the game $G_2^*(X)$ by $G^*(X)$. As hinted by the previous remark, we will prove that if X is determined, then X has the PSP.

Proposition 2.48. If P1 has a winning strategy in $G^*(X)$, then X contains a perfect set.

Proof. Let σ be a winning strategy for P1 in the game $G^*(X)$ and define the map $f : 2^\omega \mapsto X$ by setting :

$$x \mapsto \sigma(\emptyset) \frown x(0) \frown \sigma(\emptyset, x(0)) \frown x(1) \frown \sigma(\emptyset, x(0), \sigma(\emptyset, x(0)), x(1)) \frown x(2) \frown ...$$

Since σ is winning, f is well-defined. Furthermore, it is easy to check that f is continuous and injective. Since 2^ω is compact and f is injective and continuous, it follows that f is a homeomorphism between 2^ω and $f(2^\omega)$ and consequently, $f(2^\omega) \subseteq X$ is perfect. ■

Proposition 2.49. If P2 has a winning strategy in $G^*(X)$, then X is countable.

Proof. Let τ be a winning strategy for P2 in the game $G^*(X)$.

Consider a position $p = (s_0, j_0, ..., s_n, j_n)$ where P2 has followed τ and it is P1 turn to play. Let $x \in X$ and $\mu := s_0 \frown j_0 \frown ... \frown s_n \frown j_n \subseteq x$. We say that x *is rejected* by p if for all s_{n+1}, if $\mu \frown s_{n+1} \subseteq x$ then we have that $\mu \frown s_{n+1} \frown \tau((s_0, ..., s_{n+1})) \nsubseteq x$.

It is enough to prove the following claims :

Claim 1 : If $x \in X$, then there is a some p such that x is rejected by p.

Claim 2 : There is at most one $x \in X$ rejected by any p

Assuming these claims, $p \mapsto x$ is a well-defined injective map and since there are countably many positions p, we conclude that X is countable.

It remains to prove these claims :

Proof of Claim 1: Suppose that there is some $x \in X$ such that no position p rejects x. In particular, x is not rejected at the empty position and thus, there is some s_0 such that $s_0 \subseteq x$ and $s_0 \frown \tau(s_0) \subseteq x$. We define recursively a play of the game : Let $p_n = (s_0, j_0, ..., s_n, j_n)$ and let $\mu_n = s_0 \frown j_0 \frown ... \frown s_n \frown j_n$. Assume that $\mu_n \subseteq x$. Then, there is some s_{n+1} such that $\mu_n \frown s_{n+1} \frown \tau(s_0, ..., s_{n+1}) \subseteq x$. Clearly, $\bigcup_n \mu_n = x \in X$ which contradicts the fact that τ is a winning strategy for P2.

Proof of Claim 2 : Let x be rejected at $p = (s_0, j_0, ..., s_n, j_n)$. By definition, $s_0 \frown j_0 \frown ...s_n \frown j_n \subseteq x$. Let $k = |s_0 \frown ... \frown j_n|$ and $\mu = s_0 \frown ... \frown j_n$. We assume that P1 plays $s_{n+1} = \emptyset$, so that $\mu \frown s_{n+1} \frown \tau(s_0, ..., s_{n+1}) \subsetneq x$. It follows that $x(k) = 1 - \tau(s_0, ..., s_{n+1})$. Proceeding similarly, assuming that P1 played $s_{n+1} = x(k)$, we determine $x(k+1)$. Hence, and repeating this for the remaining coordinates, we conclude that x is uniquely determined by p (and by τ). ■

Corollary 2.50. If $X \subseteq 2^\omega$ is uncountable and $G^*(X)$ is determined, then X contains a perfect set.

In order to finally establish the connection between *-games and the PSP, we need the following result :

Proposition 2.51. Suppose that Γ is a pointclass of sets of ω^ω closed under continuous preimages. Then, all $A \in \Gamma$ are determined for the game $G(\omega, A)$ if and only if all $\omega^\omega \setminus A$ are also determined.

Proof. We assume that all sets in the pointclass Γ are determined and suppose that $A \subseteq \omega^\omega$ is such that $A \in \neg\Gamma$. Define $f : \omega^\omega \to \omega^\omega$ such that $f(x)(n) := x(n+1)$. Clearly, f is continuous and since Γ is closed under continuous preimages, $f^{-1}(\neg A) = \neg f^{-1}(A) \in \Gamma$ and thus, $f^{-1}(A) \in \neg\Gamma$. Let $B = \omega^\omega \setminus f^{-1}(A)$, so that B is determined as $B \in \Gamma$.
If τ is a winning strategy for P2 on $G(\omega, B)$. We can then define a winning strategy σ for P1 on $G(\omega, A)$. Indeed, set $\sigma(s) := \tau(0 \frown s)$ for all $s \in \omega^{<\omega}$ for which this is well-defined. Let $x \in [\sigma]$ so that $0 \frown x \in [\tau]$. Thus, $0 \frown x \notin B$ and we conclude that $f(0 \frown x) = x \in A$. If instead σ is a winning strategy for P1 on $G(\omega, B)$, we proceed similarly and define a winning strategy for P2 on $G(\omega, A)$. ■

Remark 2.52. As an immediate consequence, all analytic subsets of ω^ω are determined if and only if all coanalytic subsets of ω^ω are determined.

We now introduce the following axiom[6] :

$\mathbf{\Sigma^1_1}$-det : *If* $A \in \mathbf{\Sigma^1_1}$, *then the game* $G(\omega, A)$ *is determined*

Finally, we prove that under $\mathbf{\Sigma^1_1}$-determinacy, all coanalytic subsets of a Polish space verify the PSP and thus, CH holds :

[6] $\mathbf{\Sigma^1_1}$-det is independent from ZFC

Theorem 2.53. *($\mathbf{\Sigma^1_1}$-Det) All coanalytic sets of a Polish space have the PSP. In particular, the CH holds for coanalytic sets.*

Proof. Let X be a Polish space and $B \subseteq X$ an uncountable coanalytic set. By Theorem 4.74, there is a Borel isomorphism $\varphi : X \to 2^\omega$. Hence, $\varphi(B)$ is an uncountable coanalytic subset of 2^ω. By Proposition 2.51, and under $\mathbf{\Sigma^1_1}$-det, $G^*(\varphi(B))$ is determined and thus, by Corollary 2.50, it contains a perfect set P. Since P is Borel, then so is $\varphi^{-1}(P)$ and by Theorem 1.33 it contains a perfect set. ■

Remark 2.54. Under the Axiom of Determinacy, the Perfect Set Property holds for any subset of a Polish space.

2.4.2 Banach-Mazur games and Projective Determinacy

We now introduce the Banach-Mazur game. Previously, we approached some regularity properties - namely the PSP, the BP and being Lebesgue measurable. We have already seen how does the covering game relates with Lebesgue measurability and how does the *-game relates with the PSP. In particular, we have seen that under the AD every subset of the real line is Lebesgue measurable and that under $\mathbf{\Sigma^1_1}$-det, every coanalytic subset of a Polish space has the PSP. In what follows, we briefly explain how does the Banach-Mazur game relate with the BP and in this context, we introduce the Axiom of Projective Determinacy.

Suppose X is a Polish space with a countable basis $\{\mathcal{U}_i\}$ and let $A \subseteq X$. In this game, at each turn P1 chooses a basic open set $\mathcal{U}_n$ and P2 chooses a basic open set $\mathcal{V}_n$ such that $\overline{\mathcal{V}_n} \subseteq \overline{\mathcal{U}_n}$ and such that $\operatorname{diam}(\mathcal{V}_n) < 2^{-1}\operatorname{diam}(\mathcal{U}_n)$. Then, P1 wins this game if and only if $x := \bigcap_n \overline{\mathcal{U}_n} \in A$. Otherwise, P2 wins. This is the Banach-Mazur game $G^{**}(A)$. Clearly, one can encode $G^{**}(A)$ as an integer game by using the enumeration of the countable basis $\{\mathcal{U}_n\}$. Indeed, each player chooses some integer $s_i \in \omega$ at each turn which corresponds to a choice of a basic open set (for instance, P1 plays s_{2i} which corresponds to choose $\mathcal{U}_{s_{2i}}$ and P2 plays s_{2i+1} which corresponds to choose $\mathcal{U}_{s_{2i+1}}$). The first player to choose an integer such that the associated basic open set violates the rules of the game $G^{**}(A)$, loses the game. If no player violates the rules, in the end there will be an unique $x \in \bigcap_i \overline{\mathcal{U}_i}$ and P1 wins if and only if $x \in A$. Henceforth, we will still denote by $G^{**}(A)$ the associated integer game, as it will always be clear from the context.

Remark 2.55. Let $(s_i)_{i\in\omega} \in \omega^\omega$. We say that (s_i) is *admissible* if it encodes a legal play in the Banach-Mazur game $G^{**}(A)$, i.e if the sequence of basic open sets $\{\mathcal{U}_{s(i)}\}_{i\in\omega}$ is such that $\overline{\mathcal{U}_{s(i+1)}} \subseteq \overline{\mathcal{U}_{s(i)}}$ and such that $\operatorname{diam}(\mathcal{U}_{s(i+1)}) < 2^{-1}\operatorname{diam}(\mathcal{U}_{s(i)})$. Let $\mathcal{A}$ be the set of all admissible sequences, clearly a closed subset of ω^ω and define $f : \mathcal{A} \to X$ such that $f((s_i)) := \bigcap_i \overline{\mathcal{U}_{s(i)}}$, easily seen to be a continuous map. Furthermore, let Γ be a pointclass which is closed under continuous preimages. It follows that under Γ-Det, one has that for all

$A \in \Gamma(X)$, the game $G^{**}(A)$ is determined. In particular, if Γ is any projective hierarchy. This motivates the following :

Definition 2.56. The Axiom of Projective Determinacy (PD) is the following statement[7] :

If P is any projective set, the game $G(\omega, P)$ is determined.

The Banach-Mazur game is strongly related with meager sets :

Theorem 2.57. *Let X be a Polish space with a countable basis $\{\mathcal{U}_i\}_{i\in\omega}$ and suppose that $A \subseteq X$. Then, the following are true :*

(i) P2 has a winning strategy on $G^{**}(A)$ if and only if A is meager

(ii) P1 has a winning strategy on $G^{**}(A)$ if and only if there is some $s \in \omega$ such that $\overline{\mathcal{U}_s} \setminus A$ is meager.

Proof. (i) Suppose that A is meager, with $A \subseteq \bigcup_n F_n$, for F_n closed and with empty interior. We will define a winning strategy τ for P2 on $G^{**}(A)$. Let P1 play s_0. Then, P2 plays s_1 such that (s_0, s_1) is a legal position and such that $\overline{\mathcal{U}_{s_1}} \cap F_0 = \emptyset$. Note that such choice is possible since F_0 has empty interior and thus, in particular there is some $x \in \mathcal{U}_{s_0} \setminus F_0$. Hence, and since F_0 is closed, $\mathcal{U}_{s_0} \setminus F_0$ is a non-empty open set and using the fact that X is a regular space, P2 can choose such s_1. We define τ inductively as to request P2 to play s_{2n+1} such that $(s_0, ..., s_{2n+1})$ is a legal position and such that $\overline{\mathcal{U}_{s_{2n+1}}} \cap F_n = \emptyset$. Once again, such choice is possible since X is regular and each F_n is a closed set with empty interior. Clearly, this is a winning strategy for P2.
Conversely, suppose that τ is a winning strategy for P2 on $G^{**}(A)$ and let $x \in X$. For an even sized finite sequence $s = (s_0, ..., s_n)$, we will say that s is *good* if it is an initial segment of some play in $G^{**}(A)$ where P2 played by τ and $x \in \overline{\mathcal{U}_{s_n}}$. Note that if every *good* finite sequence has a *good* extension, then $x \notin A$. Hence, if $x \in A$ then there is some $(s_0, ..., s_n)$ which is maximal among *good* finite sequences (for x). Now let $s = (s_0, ..., s_n)$ be any even sized sequence and define the following set :

$$B(s_0, ..., s_n) = \cap\{\overline{\mathcal{U}_{s_n}} \setminus \mathcal{U}_{\tau(s_0,...,s_n,s)} : \overline{\mathcal{U}_s} \subseteq \overline{\mathcal{U}_{s_n}} \text{ and } \mathrm{diam}(\mathcal{U}_s) < 2^{-1}\mathrm{diam}(\mathcal{U}_{s_n})\}$$

It is easy to check that each $B(s_0, ..., s_n)$ is closed and nowhere dense. Moreover, if $x \in A$ then $x \in B(s_0, ..., s_n)$ for some $(s_0, ..., s_n)$ from where it follows that A is meager.

(ii) Suppose that $\overline{\mathcal{U}_s} \setminus A$ is meager for some $s \in \omega$. We define a winning strategy σ for P1 as follows : P1 starts by playing s. Then, P2 needs to play some t such that $\overline{\mathcal{U}_t} \subseteq \overline{\mathcal{U}_s}$. Let $\overline{\mathcal{U}_s} \setminus A \subseteq \bigcup_n F_n$ for some closed F_n with empty interior. Then, P1 replies to P2 by playing s_1 such that (s, t, s_1) is a legal position and

[7] PD is independent from ZFC

such that $\overline{\mathcal{U}_{s_1}} \cap F_0 = \emptyset$. We define σ inductively following the same reasoning as in (i).
Conversely, suppose that σ is a winning strategy for P1 on $G^{**}(A)$ such that requests P1 to start the game with $s_0 \in \omega$. Notice that P2 wins the game $G^{**}(\overline{\mathcal{U}_{s_0}} \setminus A)$ by following σ and thus, by (i), it follows that $\overline{\mathcal{U}_{s_0}} \setminus A$ is meager. ■

Corollary 2.58. Assume the PD. Then, every projective set of a Polish space has the BP.

Proof. Let $A \in \mathbf{\Sigma}^1_n(X)$, for a Polish space X with countable basis $\{\mathcal{U}_i\}$. By the PD (and Remark 2.55), $G^{**}(A)$ is determined. Hence, by Theorem 2.57, either A is meager or there is some $s \in \omega$ such that $\overline{\mathcal{U}_s} \setminus A$ is meager. Define :

$$B = \bigcup \{\mathcal{U}_s : \overline{\mathcal{U}_s} \setminus A \text{ is meager}\}$$

If we prove that $A \Delta B$ is meager, it follows that A has the BP, hence it is enough to prove that both $A \setminus B$ and $B \setminus A$ are meager.
On one hand, $B \setminus A \subset \bigcup\{\overline{\mathcal{U}_s} \setminus A : \mathcal{U}_s \in B\}$ and thus, $B \setminus A$ is meager.
On the other hand, suppose towards a contradiction that $A \setminus B$ is non meager. Note that $A \setminus B \in \mathbf{\Sigma}^1_n$ and thus, under the PD (and by Theorem 2.57), one has that there is some $s \in \omega$ such that $\overline{\mathcal{U}_s} \setminus (A \setminus B)$ is meager. Thus, $\overline{\mathcal{U}_s} \setminus A$ is also meager and hence, $\mathcal{U}_s \subseteq B$. It follows that $\mathcal{U}_s \subseteq \overline{\mathcal{U}_s} \setminus (A \setminus B)$ and thus $\mathcal{U}_s$ is meager, which contradicts the Baire Category Theorem. ■

3 Extended Examples

In this section, we classify some sets of interest according to their complexity. The section is divided as follows :

• In section 3.1, we consider important examples of $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ complete sets. These constitute a set of useful examples which are often used in order to establish continuous (Wadge) reductions and consequently identify the complexity of other sets. Indeed, each example that we study in this section shall be used in further sections.

• In Section 3.2, we follow [14] and we greatly extend a classical result due to Mazurkiewicz concerning differentiable functions $f \in C([0,1])$. We prove that the sets of elements in $C([0,1])$ that are everywhere differentiable, piecewise differentiable or differentiable on cocountable sets, are all $\mathbf{\Pi}^1_1$-complete.

• In Section 3.3, we follow [12] and [18] and prove that the family of closed sets of uniqueness is also $\mathbf{\Pi}^1_1$-complete. This is a topic of great historical and research interest and we aim as well to give a (very) brief overview on it.

• In Section 3.4, we follow [13] and prove that every bounded analytic subset of

$\mathbb{C}$ is the point spectrum of some bounded operator in a separable Banach space. Moreover, we follow [19] and prove that if X is a separable and reflexive Banach space and $T \in \mathcal{L}(X)$, then its point spectrum $\sigma_p(T)$ is Borel. Based on similar arguments, we also prove that the set $\{F \in \mathcal{F}(X) : F$ is uncountable and $F \neq \ker(T - \lambda 1)\}$ is an analytic set which is not Borel with respect to the Effros space.

3.1 Preliminaries

Let X be a Polish space and $\Gamma(X)$ a pointclass of subsets of X. Recall that $A \subseteq X$ is said to be Γ-hard if for every zero dimensional space Y and $B \in \Gamma(Y)$, there is a continuous map $f : Y \to X$ such that $f^{-1}(A) = B$, i.e $B \leq_W A$ (Definition 2.31). Furthermore, if $A \in \Gamma(X)$ then A is said to be Γ-complete. In this section, we consider the case when Γ is either $\mathbf{\Sigma^1_1}$ (analytic sets) or $\mathbf{\Pi^1_1}$ (coanalytic sets).

We start by proving that the set WF of well-founded trees (on ω) is $\mathbf{\Pi^1_1}$-complete. This fact will be used several times in the remaining of this section. In a way, it is fair to say that WF is the archetypal $\mathbf{\Pi^1_1}$-complete set.

We start by identifying a tree T on $\mathbb{N}$ with its characteristic map $\chi_T \in 2^{\omega^{<\omega}}$ and let Tr be the set of all trees. Note that :

$$T \in Tr \text{ if and only if } \forall s, t \in \omega^{<\omega} : s \in T \text{ and } t \preceq s, \text{ then } t \in T$$

It follows immediately that Tr is a G_δ set in $2^{\omega^{<\omega}}$ and thus, a Polish space.

Recall that if $T \in Tr$ is such that $[T] = \emptyset$, then T is said to be well-founded. Otherwise, T is said to be ill-founded. We let WF be the set of all well-founded trees (on ω) and IL be its complement, i.e the set of all ill-founded trees. We define :

$$E = \{(T, \beta) \in 2^{\omega^{<\omega}} \times \omega^\omega : T \in Tr \text{ and there is some } n \text{ s.t } T(\beta|_n) = 0\}$$

It is easily shown that E is a Borel set and since $WF = \forall^{\omega^\omega} E$ (i.e WF is the coprojection of E), it follows that WF is coanalytic. Now, we prove that WF is $\mathbf{\Pi^1_1}$-hard :

Let $C \subseteq \omega^\omega$ be any coanalytic set. In order to prove that WF is $\mathbf{\Pi^1_1}$-hard, we provide a continuous map $f : \omega^\omega \to Tr$ such that $f^{-1}(WF) = C$. We note that this suffices by Remark 2.16.

By Proposition 1.36, there is a tree T on $\omega \times \omega$ such that $\alpha \in C$ if and only if $T(\alpha)$ is well-founded. We let $f : \omega^\omega \to Tr$ be such that $f(\alpha) = T(\alpha)$ and it follows that $C = f^{-1}(WF)$. Hence, it remains to show that f is a continuous map : let $\mathcal{U} \in \mathcal{N}_{T(\alpha)}$ be a basic open set of the form $\{T \in Tr : T(s) = 1\}$

for some $s \in \omega^{<\omega}$ such that $f(\alpha)(s) = 1$. Note that $f(\alpha)(s) = 1$ if and only if $T(\alpha|_{|s|}, s) = 1$ so if we take $\mathcal{V} \in \mathcal{N}_\alpha$ of the form $\Sigma(\alpha|_{|s|})$, it is clear that $f(\mathcal{V}) \subseteq \mathcal{U}$.

Therefore, we have proved the following :

Theorem 3.1. *The set WF is $\mathbf{\Pi^1_1}$-complete and the set IF is $\mathbf{\Sigma^1_1}$-complete.*

In order to consider the next example of a $\mathbf{\Pi^1_1}$-complete set, we define $\mathbb{Q}' = \mathbb{Q} \cap [0,1]$ and let $\mathcal{K}(\mathbb{Q}') = \{K \in \mathcal{K}([0,1]) : K \subseteq \mathbb{Q}'\}$. This set will be of crucial importance for Section 3.3. We consider the hyperspace of compact subsets of $[0,1]$ endowed with the Vietoris topology (see Appedix, Section 4.4).

Theorem 3.2. *The set $\mathcal{K}(\mathbb{Q}')$ is $\mathbf{\Pi^1_1}$-complete.*

Proof. **Step 1** : We first prove that $\mathcal{K}(\mathbb{Q}')$ is coanalytic. In order to do so, let :

$$G = \{(K,x) : x \in K \text{ and } x \in N\} \subseteq \mathcal{K}([0,1]) \times [0,1], \text{ where } N = [0,1] \setminus \mathbb{Q}'$$

Note that :

$$\neg\mathcal{K}(\mathbb{Q}') = \{K \in \mathcal{K}([0,1]) : K \cap N \neq \emptyset\} = \pi_1(G)$$

We show that G is a G_δ set and thus, a Polish space. Since $\neg\mathcal{K}(\mathbb{Q}')$ is then a continuous image of a Polish space, it follows that $\neg\mathcal{K}(\mathbb{Q}')$ is analytic and thus, $\mathcal{K}(\mathbb{Q}')$ is coanalytic. We consider the map :

$$\pi_2 : \mathcal{K}([0,1]) \times [0,1] \to [0,1] \text{ such that } (K,x) \mapsto x$$

Note that N is G_δ and thus, so is $\pi_2^{-1}(N)$. On the other hand, $C = \{(K,x) : x \in K\}$ is closed (by Theorem 4.82 (i)). Since $G = \pi_2^{-1}(N) \cap C$, it follows that $G \in \mathbf{\Sigma^0_2}$ and we are done.

Step 2 : Let $F \subseteq 2^\omega$ be a F_σ set. We prove that there is a continuous map $g : 2^\omega \to 2^\omega$ such that $F = g^{-1}(E)$, where $E \subseteq 2^\omega$ is the set of eventually periodic sequences. Note that E is a countable and dense subset of 2^ω. Furthemore, by Proposition 2.21 (and Remark 2.16 or Theorem 4.69) one has that either $F \leq_W E$ or $E \leq_W \neg F$. Assume, towards a contradiction, that there is some continuous reduction f such that $f^{-1}(\neg F) = E$. Then, and since F is a F_σ set, it follows that E is a dense G_δ set. On the other hand, and since E is countable, it follows that its complement is also a dense G_δ set, which contradicts the Baire Category Theorem. Hence, $F \leq_W E$ as we wanted to prove.

Step 3 : Let $\Lambda = \{K \in \mathcal{K}(2^\omega) : K \subseteq E\}$, with E as in Step 2. We prove that Λ is $\mathbf{\Pi^1_1}$-complete. Indeed, similarly to Step 1 (since E is countable and thus, F_σ), one has that Λ is coanalytic. Now consider the set WF seen as a $\mathbf{\Pi^1_1}$-complete subset of 2^ω (the Cantor set is homeomorphic to any countable product of itself). In particular, it follows by Theorem 1.13 that there is a F_σ

set $B \subseteq 2^\omega \times 2^\omega$ such that :

$$x \in WF \text{ if and only if for all } \epsilon \in 2^\omega \text{ one has that } (x, \epsilon) \in B$$

By Step 2, let g be a continuous map such that $B = g^{-1}(E)$ and define a map $\Psi : 2^\omega \to \mathcal{K}(2^\omega)$ such that $x \mapsto g(\{x\} \times 2^\omega)$. It is clear that Ψ is continuous and that $x \in WF$ if and only if $\Psi(x) \in \Lambda$. By Theorem 3.1, it follows that Λ is $\mathbf{\Pi^1_1}$-complete.

Step 4 : Let $f : 2^\omega \to [0,1]$ such that $x \mapsto \sum_{n=0} \frac{x(n)}{2^{n+1}}$. Note that f is continuous and $x \in E$ if and only if $f(x) \in \mathbb{Q}'$. Now we simple define $F : \mathcal{K}(2^\omega) \to \mathcal{K}([0,1])$ by $F(K) = f(K)$. By Theorem 4.82 (v) one has that F is continuous and furthermore, $F(K) \in \mathbb{Q}'$ if and only if $K \in \mathcal{K}(E)$. By Step 3, this proves that $\mathcal{K}(\mathbb{Q}')$ is $\mathbf{\Pi^1_1}$-complete. ■

The next important example, which will be used in Section 3.4, is due to Hurewicz :

Theorem 3.3. *Let X be a Polish space. Then, $\{K \in \mathcal{K}(X) : K \text{ is uncountable}\}$ is $\mathbf{\Sigma^1_1}$ and if X is uncountable, it is $\mathbf{\Sigma^1_1}$-complete. Similarly, $\{F \in \mathcal{F}(X) : F \text{ is uncountable}\}$ is $\mathbf{\Sigma^1_1}$ and if X is uncountable, it is $\mathbf{\Sigma^1_1}$-complete.*

Proof. Note that by Theorem 4.83, the Effros Borel space of $\mathcal{F}(2^\omega)$ coincides with the Vietoris topology. We divide the proof in the following steps :

Step 1 : Let $N \subseteq 2^\omega$ be the set of all binary sequences with infinitely many 1's. Note that $\neg N$ is a countable and dense subset of 2^ω and thus, proceeding similarly as in Steps 2 and 3 of the proof of Theorem 3.2, one can prove that $\{K \in \mathcal{K}(2^\omega) : K \subseteq \neg N\}$ is $\mathbf{\Pi^1_1}$-complete. It follows that $\{K \in \mathcal{K}(2^\omega) : K \cap N \neq \emptyset\}$ is $\mathbf{\Sigma^1_1}$-complete.

Step 2 : Let $P(X)$ be the set of non-empty perfect subsets of X and let $\{V_n\}$ be a countable basis for X. Then, note that K is perfect if and only if the following condition $(*)$ holds :

$$\forall n (V_n \cap K \neq \emptyset \Rightarrow \exists k, l : V_k \cap V_l = \emptyset, V_k \cup V_l \subseteq V_n, V_l \cap K \neq \emptyset, V_k \cap K \neq \emptyset)$$

One can rewrite the condition $(*)$ using the following auxiliary sets :

$$A_n = \{K \in \mathcal{K}(X) : K \cap V_n \neq \emptyset\}$$

$$S_n = \{(k,l) : V_k, V_l \subseteq V_n, V_k \cap V_l = \emptyset\}$$

Indeed, one can prove that the following holds :

$$P(X) = \bigcap_n (\neg A_n \cup \bigcup_{(k,l) \in S_n} (A_k \cap A_l))$$

Using Theorem 4.82 and since S_n is countable, we conclude that $P(X)$ is Borel.

By Theorem 4.82, it follows that $\Omega = \{(K, L) \in \mathcal{K}(X) \times \mathcal{K}(X) : K \subseteq L\}$ is closed in $\mathcal{K}(X) \times \mathcal{K}(X)$ and thus, a Polish space. Moreover, $\{(K, L) \in P(X) \times \mathcal{K}(X) : K \subseteq L\}$ is a Borel set in Ω. On the other hand, (by Theorem 4.64 and 4.63) every uncountable Polish space contains a copy of 2^ω. Thus, for $K \in \mathcal{K}(X)$ one has that K is uncountable if and only if there is some $P \in P(X)$ such that $P \subseteq K$. It follows, by Definition 1.12, that $\{K \in \mathcal{K}(X) : K \text{ is uncountable}\}$ is analytic. Similarly, but appealing to Theorem 4.84 (instead of Theorem 4.82) we conclude that $\{F \in \mathcal{F}(X) : F \text{ is uncountable}\}$ is also analytic. Therefore, we proved that the following sets are analytic :

$$\{F \in \mathcal{F}(X) : F \text{ is uncountable}\}$$

$$\{K \in \mathcal{K}(X) : K \text{ is uncountable}\}$$

Step 3 : We first assume that $X = 2^\omega$. Define $f : 2^\omega \to \mathcal{K}(2^\omega)$ such that:

$$f(x) = \{y \in 2^\omega : y \leq x \text{ pointwise}\}$$

Then, f is continuous and such that if $x \in N$ then $f(x)$ is non-empty and perfect and if $x \notin N$ then $f(x)$ is finite. Let $g : \mathcal{K}(2^\omega) \to \mathcal{K}(2^\omega)$ such that :

$$g(K) = \bigcup_{x \in K} f(x)$$

By Theorem 4.82, g is continuous. Moreover, one has that :

$$K \cap N \neq \emptyset \text{ if and only if } g(K) \text{ is uncountable}$$

Hence, $g^{-1}(\{K \in \mathcal{K}(2^\omega) : K \text{ is uncountable}\}) = \{K \in \mathcal{K}(2^\omega) : K \cap N \neq \emptyset\}$ and by Step 1, we can conclude that $\{K \in \mathcal{K}(2^\omega) : K \text{ is uncountable}\}$ and $\{F \in \mathcal{F}(2^\omega) : F \text{ is uncountable}\}$ are $\mathbf{\Sigma^1_1}$-hard.
Finally, we note that by Theorem 4.65 any uncountable Polish space contains a homeomorph copy of 2^ω and consequently, the sets $\{K \in \mathcal{K}(X) : K \text{ is uncountable}\}$ and $\{F \in \mathcal{F}(X) : F \text{ is uncountable}\}$ are $\mathbf{\Sigma^1_1}$-hard. Since by Step 2 these sets are also analytic, we are done.

■

3.2 Differentiable functions

In this section, we follow mainly [14]. We extend a classical result of Mazurkiewicz (c.f [23]) on everywhere differentiable functions on $[0, 1]$. We consider the Polish space $C([0, 1])$ with the usual uniform norm. The argument used in [14] is a *clever* modification of Mazurkiewicz original construction, which can be also found in [1] (Theorem 33.9, p. 248). Before presenting the main argument, we need to introduce some notation:

Given a closed interval $I = [a,b] \subseteq [0,1]$ we denote its length by $|I|$ and we define a map $\varphi(x,I) : [0,1] \to [0,1]$ as follows :

$$\varphi(x,I) = \begin{cases} \frac{16(x-a)^2(x-b)^2}{(b-a)^3}, & \text{if } x \in I \\ 0, & \text{otherwise} \end{cases}$$

We note that $\varphi(x,I)$ is non-negative and attains its maximum at $x = \frac{a+b}{2}$ which is $b-a$. Moreover, for an interval $I = [a,b]$ we define $I^L = [a, \frac{a+b}{2}]$ and $I^R = [\frac{a+b}{2}, b]$.

We also define $Z = \{(s,d) \in \omega^{<\omega} \times 2^{<\omega} : |s| = |d|\}$, clearly a countable set and we fix some bijection $i : Z \to \omega$. Then, for a tree T (on ω) we define the set $Z(T) = \{(s,d) \in Z : s \in T\}$ and for any element $(s,d) \in Z(T)$, let $|(s,d)| = |s| = |d|$. Finally, and setting $J_{(\emptyset,\emptyset)} = [0,1]$, we define by induction on the length of $(s,d) \in Z$ some closed intervals $J_{(s,d)}$ and $K_{(s,d)}$ with the following properties :

(i) $K_{(s,d)} \subseteq J_{(s,d)}$, such that $|K_{(s,d)}| \leq 2^{-i((s,d))}(|J_{(s,d)}| - |K_{(s,d)}|)$ and with $K_{(s,d)}$ concentric on $J_{(s,d)}$.

(ii) $J_{(s\frown n, d\frown i)} \subseteq K^L_{(s,d)}$ for every $n \in \omega$ and $i \in 2$.

(iii) $J_{(s\frown n, d\frown i)} \cap J_{(s\frown m, d\frown j)} = \emptyset$ if $n \neq m$ and $i \neq j$.

Note that if $y \in \omega^\omega$ then the following set is homeomorphic to 2^ω (see Remark 1.32) :

$$\bigcap_n \bigcup_{d \in 2^n} J_{(y|_n, d)}$$

Thus, given a tree $T \in Tr$ one can define :

$$G_T = \bigcup_{y \in [T]} \bigcap_n \bigcup_{d \in 2^n} J_{(y|_n, d)}$$

Note that :

$$T \in WF \Leftrightarrow G_T = \emptyset \text{ and } T \notin WF \Leftrightarrow G_T \text{ contains a perfect set.}$$

Finally, given a tree T, one can define the following map :

$$F_T(x) := \sum_{(s,d) \in Z(T)} \varphi(x, K^R_{(s,d)}), \text{ for } x \in [0,1]$$

Note that $0 \leq \varphi(x, K^R_{(s,d)}) \leq |K^R_{(s,d)}|$, hence $\varphi(x, K^R_{(s,d)}) \in [0, 2^{-i((s,d))}]$. Since the uniform limit of continuous maps is a continuous map, it follows that

$F_T \in C([0,1])$ and thus, we have just defined a map :

$$\begin{aligned} \Psi : Tr &\to C([0,1]) \\ T &\mapsto F_T \end{aligned}$$

Furthermore, Ψ is continuous : Indeed, let $\epsilon > 0$ and N such that $2^{-(N-2)} < \epsilon$. Fix some $T \in Tr$ and let $S \in Tr$ be in the neighborhood of T such that $T \cap O = S \cap O$, where $O = \{s \in \omega^{<\omega} : \forall d \in 2^{<\omega} : |d| = |s| \Rightarrow i((s,d)) < N\}$. This defines an open neighborhood of T, after identifying each tree with its characteristic function. Moreover :

$$|F_T(x) - F_S(x)| \leq \sum_{(s,d) \in Z(T), i((s,d)) \geq N} \varphi(x, K^R_{(s,d)}) + \sum_{(s,d) \in Z(S), i((s,d)) \geq N} \varphi(x, K^R_{(s,d)})$$

Therefore, $||F_T - F_S||_\infty \leq \sum_{i \geq N} 2^{-i+1} < \epsilon$ and this proves that $T \mapsto F_T$ is continuous.

To summarize, we have that :

• The map $\Psi : Tr \to C([0,1])$ such that $T \mapsto F_T$, is continuous.

• $T \in WF$ if and only if $G_T = \emptyset$ and $T \notin WF$ if and only if G_T contains a non empty perfect set.

In what follows, and for $f \in C([0,1])$, let $ND(f) = \{x \in [0,1]\ f'(x)$ does not exist$\}$. With this terminology, the main result in this section is the following :

Theorem 3.4. *The map $\Psi : Tr \to C([0,1])$ is such that :*

(i) $T \in WF$ if and only if $ND(\Psi(T)) = \emptyset$.

(ii) $T \notin WF$ if and only if $ND(\Psi(T))$ contains a non empty perfect set.

Assuming Theorem 3.4, one can establish the $\mathbf{\Pi^1_1}$-hardness of the sets of interest for this section :

Corollary 3.5. Let $\mathcal{F}$ be a family of countable subsets of $[0,1]$ such that $\emptyset \in \mathcal{F}$. Then, the set $\{f \in C([0,1]) : ND(f) \in \mathcal{F}\}$ is $\mathbf{\Pi^1_1}$-hard. In particular, the following sets are $\mathbf{\Pi^1_1}$-hard :

(i) $\{f \in C([0,1]) : ND(f) = \emptyset\}$

(ii) $\{f \in C([0,1]) : ND(f)$ is finite$\}$

(iii) $\{f \in C([0,1]) : ND(f)$ is countable$\}$

Proof. Let $A = \{f \in C([0,1]) : ND(f) \in \mathcal{F}\}$. Then, Ψ is a continuous reduction such that $\Psi^{-1}(A) = WF$. Indeed, if $T \in WF$ then by Theorem 3.4 $ND(F_T) = \emptyset$ and conversely, if $T \in \Psi^{-1}(A)$, one has necessarily that $T \in WF$ otherwise, by Theorem 3.4, $ND(F_T)$ is uncountable (by Theorem 4.63). Morever, by Theorem 3.1, WF is $\mathbf{\Pi^1_1}$-complete. ■

Let us now prove Theorem 3.4 :

Proof. We start by noting that it is enough to prove that for each $x \in [0,1]$ the following holds :

$$x \notin G_T \text{ if and only if } F'_T(x) \text{ exists}$$

Let us assume first that $x \in G_T$ and we prove that $F'_T(x)$ does not exist. Since $x \in G_T$, there is some $y \in [T]$ and $d \in 2^\omega$ such that $x \in K^L_{(y|_n,d|_n)}$ for all $n \in \omega$. Let c_n be the centre of the interval $K^R_{(y|_n,d|_n)}$ and $l_n = \frac{|K^R_{(y|_n,d|_n)}|}{2}$. Note that $c_n \to x$ and that $c_n + l_n \to x$. Moreover, since $x \notin K^R_{(y|_n,d|_n)}$ we have that $F_T(x) = 0$ and similarly, for every n, $F_T(c_n + l_n) = 0$. Thus, for all n, we have that $\frac{F_T(c_n+l_n)-F_T(x)}{c_n+l_n-x} = 0$. On the other hand, $\frac{F_T(c_n)-F_T(x)}{c_n-x} \geq \frac{2l_n}{3l_n} = \frac{2}{3}$, for all n, and this is enough to conclude that $F'_T(x)$ does not exist.

Now assume that $x \notin G_T$. We will prove that $F'_T(x)$ exists :

(i) Since $x \notin G_T$, x is in at most finitely many intervals $J_{(s,d)}$. Hence, let N be such that for all $(s,d) \in Z(T)$, if $i((s,d)) \geq N$ then $x \notin J_{(s,d)}$. Fix some $(s,d) \in Z(T)$ such that $i((s,d)) \geq N$ and $h \neq 0$ such that $|h| < |J_{(s,d)}| - |K_{(s,d)}|$. Since $x + h \notin K^R_{(s,d)}$ we have that $\varphi(x, K^R_{(s,d)}) = 0$ and thus :

$$\left|\frac{\varphi(x+h,K^R_{(s,d)})-\varphi(x,K^R_{(s,d)})}{h}\right| \leq \frac{|K^R_{(s,d)}|}{|J_{(s,d)}|-|K_{(s,d)}|} \leq 2^{-i((s,d))}$$

(ii) For $n \geq N$ define $F^n_T(x) := \sum_{(s,d)\in Z(T),i((s,d))\leq n} \varphi(x, K^R_{(s,d)})$. Let $\epsilon > 0$, $n \geq N$ such that $2^{-n} < \frac{\epsilon}{3}$ and $k = \min\{|(s,d)| : (s,d) \in Z(T), i((s,d)) \geq n\}$. Furthermore, fix an element $(a,b) \in Z(T)$ such that $|(a,b)| = k$ and let $\delta' = |J_{(a,b)}| - |K_{(a,b)}|$ with $h \neq 0$ such that $|h| < \delta'$. Using (i) :

$$\left|\frac{F_T(x+h)-F_T(x)}{h} - \frac{F^n_T(x+h)-F^n_T(x)}{h}\right| \leq \sum_{(s,d)\in Z(T),i((s,d))>n} \left|\frac{\varphi(x+h,K^R_{(s,d)})-\varphi(x,K^R_{(s,d)})}{h}\right|$$

Thus, $\left|\frac{F_T(x+h)-F_T(x)}{h} - \frac{F^n_T(x+h)-F^n_T(x)}{h}\right| \leq \sum_{j=n+1}^{\infty} 2^{-j} < \frac{\epsilon}{3}$.

(iii) Clearly, being a finite sum, F^n_T is differentiable. Thus, there is some $\delta \in (0, \delta']$ such that for every h, h' with $0 < |h|, |h'| < \delta$ the following holds:

$$\left|\frac{F^n_T(x+h)-F^n_T(x)}{h} - \frac{F^n_T(x+h')-F^n_T(x)}{h'}\right| < \frac{\epsilon}{3}$$

(iv) Finally, by (ii) and (iii) we can prove that $F'_T(x)$ exists. Indeed, for every h, h' such that $0 < |h|, |h'| < \delta$ the following holds :

$$\left|\frac{F_T(x+h)-F_T(x)}{h} - \frac{F_T(x+h')-F_T(x)}{h'}\right| \leq \left|\frac{F_T(x+h)-F_T(x)}{h} - \frac{F^n_T(x+h)-F^n_T(x)}{h}\right| + \left|\frac{F^n_T(x+h)-F^n_T(x)}{h} - \frac{F^n_T(x+h')-F^n_T(x)}{h'}\right| + \left|\frac{F^n_T(x+h')-F^n_T(x)}{h'} - \frac{F_T(x+h')-F_T(x)}{h'}\right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon \qquad \blacksquare$$

Proposition 3.6. Let X and Y be standard Borel spaces and $A \subseteq X \times Y$ be analytic. Then, $\{x \in X : A_x \text{ is countable}\}$ is coanalytic.

Proof. This is known as the Mazurkiewicz-Sierpinski Theorem. The reader can find a proof in [1] (Theorem 29.19, p. 231). ■

Corollary 3.7. The following sets are $\mathbf{\Pi^1_1}$-complete :

(i) $\{f \in C([0,1]) : ND(f) = \emptyset\}$ (Mazurkiewicz)

(ii) $\{f \in C([0,1]) : ND(f)$ is finite$\}$ (Sofronidis)

(iii) $\{f \in C([0,1]) : ND(f)$ is countable$\}$

Proof. By Corollary 3.5 it is enough to prove that each of those sets is coanalytic:
(i) Note that $f \in C([0,1])$ is differentiable at some point $x \in [0,1]$ if and only if for each $n \in \omega$ there is some $m \in \omega$ such that if $0 < |h_1|, |h_2| < \frac{1}{m}$ and if $x + h_1 \in [0,1]$ and $x + h_2 \in [0,1]$, then the following holds :

$$(*)\ |\tfrac{f(x+h_1)-f(x)}{h_1} - \tfrac{f(x+h_2)-f(x)}{h_2}| \leq \tfrac{1}{n}$$

For each n, m let $E(n,m) = \{(f,x) \in C([0,1]) \times [0,1]$ such that $(*)$ holds$\}$. It is easy to check that each $E(n,m)$ is closed. Now let :

$$E = \{(f,x) \in C([0,1]) \times [0,1] :\ f'(x) \text{ does not exist}\}$$

Since E is the complement of $\bigcap_n \bigcup_m E(n,m)$, one has that E is $\mathbf{\Sigma^0_3}$ and since $\{f \in C([0,1]) : ND(f) = \emptyset\}$ is the complement of the projection of E onto $C([0,1])$, it follows that is coanalytic.

(ii) The following set can be shown to be Borel (c.f [15]) :

$$B = \{(f,(x_n)) \in C([0,1]) \times [0,1]^\omega : \forall i \neq j : x_i \neq x_j \wedge \forall n\ f'(x_n) \text{ does not exist}\}$$

The set of piece-wise differentiable functions is simply the complement of the projection of B onto $C([0,1])$.

(iii) Let $E_f = \{x \in [0,1] : (x,f) \in E\}$. Clearly, the set of continuous functions on $[0,1]$ which are differentiable on a cocountable set coincides with $\{f \in C([0,1]) : E_f$ is countable$\}$. We get the desired result by simply applying Proposition 3.6 to E. ■

Remark 3.8. It is known that the set of continuous functions of $C([0,1])$ which are nowhere differentiable, is a $\mathbf{\Pi^1_1}$-complete set (cf [16]).

3.3 Sets of Uniqueness

3.3.1 Overview on sets of uniqueness

In this section we follow mainly [18] and [12]. We aim to provide a brief overview on sets of uniqueness and to classify the collection of closed sets of uniqueness

(as a set of subsets of $\mathcal{K}(\mathbb{T})$, endowed with the Vietoris topology). We ommit the proof of several classical results in order to keep this overview short and hopefuly, more appealing to the reader.
The study of trigonometric series has a very long and rich history. From Riemann to Cohen, the study of trigonometric series revealed to be a landscape of surprises that one could even argue that lead to the creation of Set Theory (or, more modestly, to the creation of the theory of ordinal numbers by Cantor). This research topic not only lead to the development of useful tools that are present in mainstream mathematics but also reveals to be an extraordinary *melting pot* of techniques. In this section, we consider some applications of descriptive set theory to it.

We start by considering formal trigonometric series of the form :

$$S(x) \sim \sum_{n=-\infty}^{+\infty} c_n e^{inx}, \text{ for } c_n \in \mathbb{C}, x \in \mathbb{R}$$

We view this as a formal expression, without any claims about its convergence at any point x. We say that a fuction $f : \mathbb{R} \to \mathbb{C}$ has a trigonometric expansion if there is some trigonometric series S which converges for every $x \in \mathbb{R}$ and such that the sum coincides with f, i.e $S(x) = f(x)$ for every $x \in \mathbb{R}$. Clearly, such f is necessarily 2π-periodic and thus, we will henceforth identify $\mathbb{T} = \{e^{ix} : 0 \leq x \leq 2\pi\}$ with $\mathbb{R}/2\pi\mathbb{Z}$ via $x \mapsto e^{ix}$ and think of $\mathbb{T}$ as $[0, 2\pi)$ (or $[0, 2\pi]$, with 0 and 2π identified).

If f is 2π-periodic and integrable, we define its Fourier coefficients as usual:

$$\hat{f}(n) = \tfrac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt$$

Uniqueness Problem : *If $S(x) = 0$ for all $x \in \mathbb{R}$ is it the case that $c_n = 0$ for all $n \in \mathbb{Z}$?*

Definition 3.9. Let $E \subseteq \mathbb{T}$. We say that E is a **set of uniqueness** if every trigonometric series $S(x) \sim \sum c_n e^{inx}$ such that $S(x) = 0$ for $e^{ix} \notin E$ (written as $x \notin E$) is identically zero. If E is not a set of uniqueness, then it is called a **set of multiplicity**.

Definition 3.10. We define $\mathcal{U} \subseteq \mathcal{K}(\mathbb{T})$ to be the set of closed sets of uniqueness (with respect to the Vietoris topology in $\mathcal{K}(\mathbb{T})$).

We will prove that $\mathcal{U}$ is $\mathbf{\Pi_1^1}$-complete. Once again, for the sake of coherence with the main topics of this thesis (and readability), we ommit the proof of several classical results that would certainly deserve a section of their own. In what follows, we try to describe the context in which the classification of sets of uniqueness is pertinent.

The first answer to the Uniqueness Problem was due to Cantor :

Theorem 3.11. *Assume $\sum c_n e^{inx} = 0$ for all but finitely many $x \in \mathbb{R}$. Then, $c_n = 0$ for all $n \in \mathbb{Z}$.*

In order to prove Theorem 3.11 we will use some classical results. We start with associating a continuous function $F_S(x)$ on $\mathbb{R}$ to any series $S(x)$ with bounded coefficients. Indeed :

Given a series $S(x) \sim \sum c_n e^{inx}$ we can formally integrate it twice and if $\{c_n\}$ is a bounded sequence, the result is absolutely and uniformly convergent so that we get a continuous function on $\mathbb{R}$:

$$F_S(x) = \tfrac{c_0 x^2}{2} - \sum_{n \in \mathbb{Z} \setminus \{0\}} \tfrac{1}{n^2} c_n e^{inx}, \text{ for } x \in \mathbb{R}$$

Given a function $F : \mathbb{R} \to \mathbb{C}$, we also define :

$$\Delta^2 F(x,h) = F(x+h) + F(x-h) - 2F(x)$$

$$D^2 F(x) = \lim_{h \to 0} \tfrac{\Delta^2 F(x,h)}{h^2}$$

Proposition 3.12. (Riemann's First Lemma) Let $S \sim \sum c_n e^{inx}$ with $\{c_n\}$ bounded. Then, if $s = \sum c_n e^{inx}$ exists, $D^2 F_S(x) = s$.

Proof. The reader can find a proof in [18] (Lemma 2.2, p. 6). ■

Proposition 3.13. (Riemann's Second Lemma) Let $S \sim c_n e^{inx}$ with $c_n \to 0$. Then, $\frac{\Delta^2 F_S(x,h)}{h} \to 0$ so if the right and left derivatives of F_S exist at some point, they must coincide.

Proof. The reader can find a proof in [18] (Lemma 2.6, p. 7). ■

Proposition 3.14. (Cantor-Lebesgue Lemma) If $\sum c_n e^{inx} = 0$ for all x in a set of positive measure, then $c_n \to 0$.

Proof. The reader can find a proof in [18] (Lemma 3.1, p.8). ■

Proposition 3.15. (Schwartz) If $F : (a,b) \to \mathbb{C}$ is continuous and $D^2 F(x) = 0$ for all $x \in (a,b)$, then F is linear on (a,b).

Proof. The reader can find a proof in [18] (Lemma 3.3, p. 9). ■

We can now return to the proof of Theorem 3.11 :

Proof. (of Theorem 3.11) : (i) First, suppose that $\sum c_n e^{inx} = 0$ for all $x \in [0, 2\pi]$. By Proposition 3.14, $c_n \to 0$ and so, in particular, $\{c_n\}$ are bounded. It follows that by Proposition 3.12, $D^2 F_S(x) = 0$ for all $x \in [0, 2\pi]$ and thus, by Proposition 3.15, F_S is linear, say $F_S(x) = ax + b$. Plugging in $x = -\pi$ and $x = \pi$ one gets that $a = 0$ and plugging in $x = 0$ and $x = 2\pi$ we get that $c_0 = 0$ so we can conclude that :

$$-\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2}c_n e^{inx} = b$$

Thus, for $m \neq 0$ we get that :

$$-\sum_{n\neq 0} \frac{1}{n^2}c_n \int_0^{2\pi} e^{i(n-m)x}dx = \int_0^{2\pi} be^{-imx}dx = 0$$

Hence, for $m \neq 0$ we have that $\frac{c_m}{m^2} = 0$ and we can conclude that $c_n = 0$ for all $n \in \mathbb{Z}$.

(ii) Now suppose that $\sum c_n e^{inx} = 0$ for $x \in [0, 2\pi] \setminus \{x_0, ..., x_m\}$ and let $0 = x_0 \leq x_1 < ... < x_{n-1} < 2\pi = x_n$. By Proposition 3.15, it follows that F_S is linear in each interval (x_i, x_{i+1}) and by Proposition 3.13 we can conclude that F_S is linear in the whole interval $[0, 2\pi]$, so that we can proceed as in (i) and conclude that $c_n = 0$ for all $n \in \mathbb{Z}$. ■

Remark 3.16. It follows from Theorem 3.11 that finite sets are sets of uniqueness. The next result extends this result to countable closed sets. We will see that sets of uniqueness can even be (uncountable) perfect sets. However, Theorem 3.17 has historical importance as it could be argued that in some sense, its original proof lead (or at least motivated) Cantor to start Set Theory (more concretely, the study of ordinal and cardinal numbers). Based on this historical significance (and admittedly due to a personal view regarding the *metaphysical* aspects of the era that was started in mathematics with the advent of Cantor's ideas) we decide to include it. It is worth to emphatize that in section 4.4 (Corollary 4.87) we generalize Theorem 3.17 under certain set-theoretical assumptions.

As previously mentioned, the next result has historical significance as it motivated Cantor to explore and formalize ideas that lead to the birth of Set Theory, more concretely the theory of ordinal numbers. We will use the usual notation for the Cantor-Bendixson derivative of a set E, i.e E^α. If needed, the reader can recall all the relevant definitions at the end of section 4.2.

Theorem 3.17. *Every closed countable set is a set of uniqueness.*

Proof. Let E be a countable closed set such that $\sum c_n e^{inx} = 0$ for $x \notin E$. Assume, without loss of generality, that $0 \notin E$. The idea is to prove by transfinite induction that F_S is linear on each contiguous interval of E^α (note that any open set of $(0, 2\pi)$ is a disjoint union of open intervals). By Theorem 4.78, it follows that F_S is linear on $(0, 2\pi)$ and as in the proof of Theorem 3.11, we can conclude that $c_n = 0$ for all $n \in \mathbb{Z}$:

(i) By definition, $E^0 = E$ and $\sum c_n e^{inx} = 0$ for $x \notin E$. Hence, it follows by Proposition 3.12 and 3.15 that F_S is linear on $\mathbb{T} \setminus E$.

(ii) Suppose that the induction hypothesis holds for α and thus, F_S is linear on each contiguous interval of E^α. Let (a, b) be a contiguous interval of

$E^{\alpha+1}$. Note that in each closed $[c,d] \subseteq (a,b)$ there are only finitely many points $c \leq x_0 < ... < x_n \leq d$ of E^α. Indeed, recall that if $L \subseteq K \subseteq \mathbb{R}$ with K compact and L infinite, then L has necessarily a limit point in K. Hence, if $[c,d] \cap E^\alpha$ is infinite, there is a limit point x of E^α in $[c,d]$ which is impossible since $x \in E^{\alpha+1}$ and $(a,b) \subseteq \mathbb{T} \setminus E^{\alpha+1}$. It follows that, by induction, F_S is linear in the intervals $(a,x_0), ..., (x_n,d)$ and consequently, by Proposition 3.13, F_S is linear in $[c,d]$. Noting that $[c,d]$ was arbitrary and that $(a,b) = \bigcup_n [a+\frac{1}{n}, b-\frac{1}{n}]$, we conclude that F_S is linear in (a,b).

(iii) Finally, let β be a limit ordinal and suppose that the hypothesis hold for all $\alpha < \beta$. Let (a,b) be a contiguous interval of E^β and consider a closed subset $[c,d] \subseteq (a,b)$. Since $E^\beta = \bigcap_{\alpha<\beta} E^\alpha$, we have that $[c,d] \subseteq \bigcup_{\alpha<\beta}((0,2\pi) \setminus E^\alpha)$. Moreover, since $[c,d]$ is compact :

$$[c,d] \subseteq \bigcup_{i \leq n}((0,2\pi) \setminus E^{\alpha_i}) \subseteq (0,2\pi) \setminus E^{\alpha_0}, \text{ for } \alpha_i \leq \alpha_0 < \beta$$

Hence, $[c,d]$ is contained in contiguous intervals of E^β and we simply apply the induction hypothesis. ∎

Remark 3.18. It should be noted that the statement of Theorem 3.17 was extended to ***any* countable set** (closed or not) already in 1908 (Bernstein) and 1909 (Young). For a survey on the chronology of the uniquess problem (and other problems concerning the characterization of sets of uniqueness) the reader is highly advised to read [24]. Again, it is worth to mention that we generalize even further this statement (under some set-theoretical assumptions) in Corollary 4.87. More concretely, working within $ZFC + \neg CH + MA(\kappa)$, then if $\aleph_0 \leq \kappa < 2^{\aleph_0}$ and $E \subset \mathbb{T}$ is such that $|E| = \kappa$, it follows that E is a set of uniqueness.

So far we have seen that sets of uniquess can be infinite and countable. In fact, and perhaps surprisingly, a set of uniquess can even be a perfect set :

Definition 3.19. A subset $E \subseteq \mathbb{T}$ is a H-set if for some non empty open interval $I \subseteq \mathbb{T}$ and some sequence $0 \leq n_0 < n_1 < ...$ we have that $(n_k E) \cap I = \emptyset$ for all k.

Remark 3.20. Every finite set is a H-set. Moreover, the Cantor set in $[0, 2\pi]$ (essentially the numbers of the form $2\pi \sum \frac{\epsilon_n}{3^n}$ with $\epsilon \in \{0,2\}$) is a H-set, as $3^n E$ avoids the middle $\frac{1}{3}$-interval.

Theorem 3.21. *Every H-set is a set of uniqueness. In particular, the Cantor set is an uncountable perfect element of* $\mathcal{U}$.

In order to prove Theorem 3.21, we need an auxiliary result from Rajchman multiplication theory. Suppose that $S \sim \sum c_n e^{inx}$ has bounded coefficients $\{c_n\}$. Let $f \in C(\mathbb{T})$ have absolutely convergent Fourier coefficients so that $f(x) = \sum \hat{f}_n e^{inx}$ converges uniformly. We define the following formal trigonometric series :

$$S(f).S \sim \sum C_n e^{inx}, \text{ with } C_n = \sum_k c_k \hat{f}(n-k)$$

Then, the following holds :

Theorem 3.22. *If* $\sum_{l=0}^{\infty} \sum_{|n|\geq l} |\hat{f}(n)| < \infty$ *and* $c_n \to 0$*, then* $\sum_{-N}^{N} C_n e^{inx} - f(x) \sum_{-N}^{N} c_n e^{inx}$ *converges to* 0 *uniformly on* x*.*

Proof. The reader can find a proof in [18] (Lemma 13.2, p. 32). ∎

Proof. (of Theorem 3.21) : (i) First, note that if E is a H-set, then so is $\overline{E}$. Indeed, and towards a contradiction, suppose that for every interval I and increasing sequence $\{n_k\}$ there is some k_0 such that $n_{k_0}\overline{E} \cap I \neq \emptyset$. We can then pick some $z \in n_{k_0}\overline{E} \cap I$. Note that for any open neighborhood $\mathcal{U}$ of z one has that $\mathcal{U} \cap n_{k_0}E \neq \emptyset$ and thus, in particular, if $\mathcal{U} \subseteq I$ we conclude that $n_{k_0}E \cap I \neq \emptyset$ which contradicts the fact that E is a H-set. Thus, and without loss of generality, we can assume that E is closed and hence, measurable.

(ii) Assume that $\sum c_n e^{inx} = 0$ for $x \notin E$ and let I be some interval and $\{n_k\}$ some sequence such that $n_k E \cap I = \emptyset$ for all k. Since E is measurable, it follows by Proposition 3.14 that $c_n \to 0$. Moreover, choose some smooth $f \in C^\infty(\mathbb{T})$ such that $\hat{f}(0) = 1$ and $\text{supp}(f) = \overline{\{x : f(x) \neq 0\}} \subseteq I$ (note that such f exists - for instance, use a bump function). Finally, define $f_k(x) := f(n_k x)$ and note that $f_k(E) = 0$ since $n_k E \cap I = \emptyset$. Furthermore, note that $f(x) = \sum \hat{f}(n)e^{inx}$ converges uniformly by classic Fourier theory.

(iii) Note that $f_k(x) = \sum \hat{f}(n)e^{in.n_k x}$ and thus :

$$\hat{f}_k(i) = \begin{cases} \hat{f}(n) & \text{if } i = n.n_k \\ 0 & \text{otherwise} \end{cases}$$

Hence, there is some $C < \infty$ such that $|\sum_{i\in\mathbb{Z}} \hat{f}_k(i)| \leq C$ for all k. Moreover, $\hat{f}_k(0) = 1$ and $\lim_{k\to\infty} \hat{f}_k(i) = 0$, for $i \neq 0$. Let :

$$C_n^k = \sum_m c_{n-m}\hat{f}_k(m) = \sum_{|m|\leq N} c_{n-m}\hat{f}_k(m) + \sum_{|m|>N} c_{n-m}\hat{f}_k(m)$$

The first term goes to c_n when $k \to \infty$ and the second term is bounded by $\sup_k\{|c_k| : |k| \geq N - |n|\}.C$. Since $c_n \to 0$, we conclude that the second term goes to zero when $k \to \infty$. Therefore, for each n we have that $C_n^k \to c_n$ as $k \to \infty$.

(iv) We now consider $S(f_k).S \sim \sum C_n^k e^{inx}$. By (ii), one has that $c_n \to 0$ and since f was chosen to be smooth, it follows by Theorem 3.22 that $\sum_{n=-N}^{N} C_n^k e^{inx} -$

$f_k(x) \sum_{n=-N}^{N} c_n e^{inx} \to 0$ for all x. Since $f_k(x) = 0$ on E and $\sum c_n e^{inx} = 0$ for $x \notin E$, it follows that $\sum_{-\infty}^{\infty} C_n^k e^{inx} = 0$ for all x and thus, it follows that $C_n^k = 0$. We conclude that, by (iii), $c_n = 0$. ∎

Hence, from the point of view of cardinality, sets of uniqueness can be very big. Indeed, and since the CH holds for closed subsets of $\mathbb{R}$, closed sets of uniqueness can be as big as possible from this point of view. However, there is a strong measure theoretical restriction to the size of a set of uniqueness. The proof relies on the following important result :

Proposition 3.23. Let f be an integrable function on $\mathbb{T}$. Then, the Fourier series of f converges to 0 on any open interval in which f vanishes.

Proof. The reader can find a proof in [18] (Lemma 7.2, p. 14). ∎

Corollary 3.24. Let $E \subseteq \mathbb{T}$ be a measurable set of uniqueness. Then, E is null.

Proof. Suppose that $\mu(E) > 0$ so that there is some closed set $F \subseteq E$ such that $\mu(F) > 0$. Let χ_F be the characteristic function of F and let $S(\chi_F)$ be its Fourier series. By Proposition 3.23, $S(\chi_F)$ converges to 0 in $\mathbb{T} \setminus F$ and since F is a set of uniqueness (as $F \subseteq E$) it follows that $\hat{\chi_F}(n) = 0$ for all n. In particular, $\hat{\chi_F}(0) = \mu(F) = 0$, which leads to a contradiction. ∎

Remark 3.25. By Corollary 3.24, all measurable sets of uniqueness have measure zero. One could ask about the converse. It seems reasonable that if $\sum c_n e^{inx}$ converges to 0 almost everywhere then it is identically zero. However, this is false and thus, not every measurable null set is a set of uniqueness. Another natural question, and since sets of uniqueness are negligible from the measure theoretic point of view, is to ask whether or not sets of uniqueness are topologically negligible. This is the :

Category problem : Is every set of uniqueness with the BP, also meager (of first category) ?

Once again, these problems are of historical significance. We shall sketch a solution to both of these questions using fairly *recent technology*, namely the Debs-Saint Raymond Theorem, which has a wonderful proof with a strong functional analytic taste. It certainly would deserve a section for itself. However, the interested reader can check [18] and [12] instead.

Theorem 3.26. *(Debs-Saint Raymond) : Let $A \subseteq \mathbb{T}$ be non-meager and with the BP. Then, there is a Borel probability measure λ on $\mathbb{T}$ such that $\lambda(A) = 1$ and $\hat{\lambda}(n) \to 0$, as $|n| \to \infty$.*[8]

[8]Recall that $\hat{\lambda}(n) := \int e^{-int} d\lambda(t)$

Proof. The reader can find a proof in [18] (Theorem 9.2, p. 18). ■

Another important result that will also be used further in this Section, is the following :

Theorem 3.27. *Let $E \subseteq \mathbb{T}$ be a closed set such that $E \neq \mathbb{T}$ and let λ be a Borel probability measure on $\mathbb{T}$ such that $\lambda(E) = 1$. Then, the following are equivalent:*

(i) $\hat{\lambda}(n) \to 0$

(ii) $\sum \hat{\lambda}(n)e^{inx} = 0$*, for all* $x \notin E$

Proof. The reader can find a proof in [18] (Theorem 7.6, p. 16). ■

Corollary 3.28. Every set of uniqueness with the BP is meager.

Proof. Let $E \subseteq \mathbb{T}$ be a set of uniqueness with the BP. Suppose, towards contradiction, that E is non-meager. Then, by Theorem 3.26 there is a Borel probability measure λ such that $\lambda(E) = 1$ and $\hat{\lambda}(n) \to 0$. In particular, there is some closed set $F \subseteq E$ such that $\lambda(F) > 0$. Let $\nu := \mu|_F$, i.e $\nu(X) = \mu(X \cap F)$.[9] One can prove that $\hat{\nu}(n) \to 0$ as $|n| \to \infty$ as well. Hence, it follows by Theorem 3.27 that $\sum \hat{\nu}(n)e^{inx} = 0$ for $x \notin F$ (and thus, for $x \notin E$). Since $\hat{v}(0) \neq 0$, it follows that E is not a set of uniqueness, leading to a contradiction. ■

Corollary 3.29. There are null sets which are not sets of uniqueness.

Proof. Take any G which is a dense G_δ (and thus, comeager) and null, i.e $\mu(G) = 0$.[10] By Baire Category Theorem, G is non-meager and since G is G_δ, then G has the BP. Thus, by Theorem 3.26, there is a Borel probability measure λ on $\mathbb{T}$ such that $\lambda(G) = 1$ and such that $\hat{\lambda}(n) \to 0$. Since $\lambda(G) > 0$, then there is some closed set $F \subseteq G$ such that $\lambda(F) > 0$. Arguing as in the proof of Corollary 3.28, and by Theorem 3.27, we can assume that $\sum \hat{\lambda}(n)e^{inx} = 0$ for $x \notin F$ (and thus, for $x \notin G$). Hence, $\sum \hat{\lambda}(n)e^{inx}$ converges to zero almost everywhere, since $\mu(G) = 0$. On the other hand, $\sum \hat{\lambda}(n)e^{inx}$ is not identically zero, since λ is a probability measure. ■

Remark 3.30. So far we have seen that any countable set is a set of uniqueness (even though we just proved it for closed countable sets) and that any set of uniqueness is null. Furthermore, we have seen that sets of uniqueness with the BP are also topologically small and that not every null set is a set of uniqueness. Moreover, we have seen that perfect sets can also be sets of uniqueness. It appears to be difficult to characterize exactly when some $E \subseteq \mathbb{T}$ is a set of uniqueness. Perhaps surprisingly there is a characterization of a type of perfect sets that allow us to identify whether or not they are a set of uniqueness. This is

[9] In all rigour, and since we apply Theorem 3.27, we should normalize ν such that $\nu(F) = 1$.

[10] Let $\{d_n\}$ be a countable and dense subset of $\mathbb{T}$ and for all k let $I_{k,n}$ be an open interval containing d_n and such that $\mu(I_{k,n}) \leq \frac{2^{-n}}{k}$. Let $G = \bigcap_k \bigcup_n I_{k,n}$.

the so called General Salem-Zygmund Theorem and it reveals a quite interesting relationship between number theory and the theory of trigonometric series. We will use it when we determine the complexity of $\mathcal{U}$.

In order to understand the statement of the General Salem-Zygmund Theorem, we shall introduce a way of producing perfect sets. First, we start with some set of parameters $\{\eta_i\}_{i=0}^{k+1}$ such that $0 = \eta_0 < \eta_1 < ... < \eta_k < \eta_{k+1} = 1$. We let $\zeta := 1 - \eta_k$ and assume that $\zeta < \eta_{i+1} - \eta_i$ for all $i < k$.

Given an interval $[a, b]$, let $l = b - a$ and consider $[a + l\eta_i, a + l\eta_i + l\zeta]$, for $0 \leq i \leq k$, all pairwise disjoint intervals. Let E be their union. We say that E is *obtained from* $[a, b]$ *by a dissection of type* $(\zeta, \eta_1, ..., \eta_k)$.

Finally, start with $E_0 = [0, 2\pi]$ and construct $E_1 \supseteq ... \supseteq E_n \supseteq ...$, where each E_{i+1} is obtained from E_i by a dissection of type $(\zeta, \eta_1, ..., \eta_k)$. We define the following perfect set :

$$E(\zeta, \eta_1, ..., \eta_k) := \bigcap_n E_n$$

The reader can check that the usual Cantor set is simply $E(\frac{1}{3}, \frac{2}{3})$.

As mentioned in Remark 3.30, there is indeed a relationship between these perfect sets, sets of uniqueness, and number theory. With this in mind, we shall introduce the concept of Pisot number.

Recall that θ is said to be an algebraic integer if it is the root of some monic polynomial $p(x) \in \mathbb{Z}[x]$. In this case, there is an unique monic polynomial $p(x)$ of least degree such that $p(\theta) = 0$. Assuming this polynomial has degree $n \geq 1$ with roots $\theta = \theta_1, ..., \theta_n$, each θ_i $(i \geq 2)$ is said to be a conjugate of θ.

Definition 3.31. An algebraic integer θ is said to be a Pisot number if $\theta > 1$ and all its conjugates have absolute value smaller than 1.

We are now in conditions to state the General Salem-Zygmund Theorem :

Theorem 3.32. *The set* $E(\zeta, \eta_1, ..., \eta_k)$ *is a set of uniqueness if and only if* $\theta := \frac{1}{\zeta}$ *is a Pisot number and* $\eta_i \in \mathbb{Q}(\theta)$*, for all* $1 \leq i \leq k$.

Proof. The reader can check [12]. ■

With our initial goal in mind of finding the complexity of $\mathcal{U}$, we can now take the final steps. A crucial missing ingredient in order to prove that $\mathcal{U}$ is $\mathbf{\Pi^1_1}$-complete in $\mathcal{K}(\mathbb{T})$ is the following :

Theorem 3.33. *The collection* $\mathcal{U} \subseteq \mathcal{K}(\mathbb{T})$ *of closed set of uniqueness is closed under countable unions.*

Proof. This is a particular case of Theorem 4.86 (see Corollary 4.96). The reader can also find a proof in [18] (Theorem 20.1, p. 48). ■

3.3.2 Complexity of $\mathcal{U}$

Relying on Theorem 3.32 and 3.33, we start this section with the following continuous reduction :

Theorem 3.34. *There is a continuous reduction of $\mathcal{U}$ to $\mathcal{K}(\mathbb{Q}')$.*

Proof. Let $f : [0,1] \to \mathcal{K}(\mathbb{T})$ such that $x \mapsto E(\frac{1}{4}, \frac{3}{8} + \frac{x}{9}, \frac{3}{4})$, which is a continuous map. Moreover, by Theorem 3.32, $x \in \mathbb{Q}$ if and only if $f(x) \in \mathcal{U}$ (indeed, $\theta = 4$ is trivially a Pisot number). Now let $F : \mathcal{K}([0,1]) \to \mathcal{K}(\mathbb{T})$ such that :

$$F(K) := \bigcup \{f(x) : x \in K\}$$

This map is continuous by Theorem 4.82. Furthermore, $K \subseteq \mathbb{Q}$ if and only if $F(K) \in \mathcal{U}$: indeed, if $K \subseteq \mathbb{Q}$ then $F(K)$ is a countable union of elements in $\mathcal{U}$ and thus, by Theorem 3.33, $F(K) \in \mathcal{U}$. On the other hand, suppose that $F(K) \in \mathcal{U}$. If $K \nsubseteq \mathbb{Q}$, there is some $x \in K \setminus \mathbb{Q}$ and thus, $f(x) \notin \mathcal{U}$. Note that a subset of a set of uniqueness is also a set of uniqueness and thus, since $F(K)$ is a set of uniqueness it would follow that so is $f(x)$, which is false. It follows that $F^{-1}(\mathcal{U}) = \mathcal{K}(\mathbb{Q}')$. ∎

Note that by Theorem 3.2 and Theorem 3.34, it **remains to prove that $\mathcal{U}$ is coanalytic in $\mathcal{K}(\mathbb{T})$**. In order to do so, we shall set up a more convenient (and quite functional analytic flavoured) context.

Recall that we can identify $(l^1)^*$ with l^∞ : given an element $\lambda = (\lambda_n) \in l^\infty$ we can view it as acting on $x = (x_n) \in l^1$ by $\lambda(x) = \sum \lambda_n x_n$. Indeed, this defines a bijective correspondence between $(l^1)^*$ and l^∞.

Recall as well that the set of complex Borel measures on $\mathbb{T}$, $\mathcal{M}(\mathbb{T})$, has a Banach space structure under which, by the Riesz-Markov Representation Theorem, can be identified with $C(\mathbb{T})^*$, by $\mu \mapsto F_\mu(f) := \int_{\mathbb{T}} f(x) d\mu(x)$.

We note that until this point, μ always denoted by default the Lebesgue measure on $\mathbb{R}$. However, for the remaining part of this section, we shall use that notation for an element $\mu \in \mathcal{M}(\mathbb{T})$.

Given a trigonometric series $S \sim \sum c_n e^{inx}$ with $\sup_n |c_n| < \infty$, we shall identify it with the element $(c_n) \in l^\infty = (l^1)^*$. We write $S(n) = c_n$. Moreover, given an element $(\lambda_n) \in l^1$, we identify it with $f(x) = \sum \lambda_n e^{inx} \in A(\mathbb{T})$, where $A(\mathbb{T})$ is the set of functions with absolutely convergent Fourier series. Note that $\hat{f}(n) = \lambda_n$ and that, for instance, e_n is identified with $f(x) = e^{inx}$.
Thus, if $S \in l^\infty$ and $f \in A(\mathbb{T})$ such that $f(x) = \sum c_n e^{inx}$, we will write :

$$\langle f, S \rangle := \textstyle\sum_n \hat{f}(n) S(-n)$$

Hence, for instance, $\langle e_n, S \rangle = S(-n)$.

Finally, if $\mu \in \mathcal{M}(\mathbb{T})$ and $S = \hat{\mu} \in l^\infty$ (i.e $\hat{\mu}(n) = \int e^{-int} d\mu$), it follows that $S \in l^\infty$ since $|\hat{\mu}(n)| < \infty$. Then, if $f \in A(\mathbb{T})$ we have :

$$\langle f, \hat{\mu} \rangle = \sum \hat{f}(n) S(-n) = \sum \hat{f}(n) \int e^{int} d\mu = \int \sum \hat{f}(n) e^{int} d\mu = \int f d\mu$$

Thus, we can view $\langle f, S \rangle$ as an integral and S as a generalized measure operating on $f \in A(\mathbb{T})$. Motivated by this analogy, we shall further refer to elements of l^∞ as pseudomeasure, and we denote $l^\infty = PM$.

Moreover, for $f \in L^1(\mathbb{T})$ we have that $\hat{f} \in c_0$, so we will refer to elements in c_0 as pseudofunctions and denote $c_0 = PF$.

The following definition will allow us to translate the concept of set of uniqueness into this functional analytic context :

Definition 3.35. Let $S \in PM$ and $K \in \mathcal{K}(\mathbb{T})$. We say that K supports S if and only if for any open interval I such that $I \cap K = \emptyset$ and any $\varphi \in C^\infty(\mathbb{T})$ such that $supp(\varphi) \subseteq I$, then $\langle \varphi, S \rangle = 0$.

We have the following generalization of Theorem 3.27 :

Theorem 3.36. *Let $K \in \mathcal{K}(\mathbb{T})$ and $S \in PF$. The following are equivalent :*

(i) S is supported by K

(ii) $\sum S(n) e^{inx} = 0$, for $x \notin K$

Proof. We will use the following (c.f [18], Theorem 12.1, p. 30):

Proposition 3.37. (Riemann's Localization Principle) Let $S \sim \sum c_n e^{inx}$ such that $c_n \to 0$. If F_S is linear on some open interval, then $S = 0$ in that interval.

$(i) \Rightarrow (ii)$: It is enough to show that F_S is linear on any open interval I which is disjoint from K. Fix some $a \in \mathbb{R}$ and $h \in (0, \pi)$. Let $\Psi_{a,h}$ be the 2π-periodic function defined on $[a - \pi, a + \pi]$ as follows :

$$\Psi_{a,h}(x) = \begin{cases} \frac{2\pi}{h} & \text{if } x = a \\ \frac{2x}{h} + \frac{2(\pi - a)}{h} & \text{if } x \in [a - \pi, a] \\ \frac{-2x}{h} + \frac{2(a + \pi)}{h} & \text{if } x \in [a, a + \pi] \end{cases}$$

We choose $a \in I$ and h small enough so that $\Psi_{a,h}$ is supported by I. Then, since S is supported by K, $\langle \Psi_{a,h}, S \rangle = 0$. We note that the Fourier series of $\Psi_{a,h}$ is given by $\sum\limits_{-\infty}^{\infty} e^{-ina} (\frac{\sin(nh/2)}{nh/2})^2 e^{inx}$ and thus :

$$\int \Psi_{a,h} d\mu = \sum \widehat{\Psi_{a,h}}(-n) \hat{\mu}(n) = \sum \hat{\mu}(n) (\tfrac{\sin(nh/2)}{nh/2})^2 e^{ina} = \tfrac{\Delta^2 F_S(a,h)}{h^2}$$

The first equality follows from the fact that if $\mu \in \mathcal{M}(\mathbb{T})$ and $f \in C(\mathbb{T})$ such that $\sum |\hat{f}(n)| < \infty$, then $\int f d\mu = \sum \hat{f}(-n)\hat{\mu}(n)$. The last equality is due to tedious computations from the definitions. We conclude that F_S is linear on I since :

$$0 = \langle \Psi_{a,h}, S\rangle = \sum \widehat{\Psi_{a,h}}(-n)S(n) = \frac{\Delta^2 F_S(a,h)}{h^2}$$

$(ii) \Rightarrow (i)$: Let $I \cap K = \emptyset$ and $\varphi \in C^\infty(\mathbb{T})$ which is supported by I. We consider the formal product $T = S(\varphi).S$, where $T(m) = \sum \hat{\varphi}(n)S(m-n)$. Since $T(0) = \sum \hat{\varphi}(n)S(-n) = \langle \varphi, S\rangle$, it is enough to prove that $T = 0$. By Theorem 3.22, we have that $\sum (T(n) - \varphi(n)S(n))e^{inx} = 0$.[11]
Since $\sum S(n)e^{inx} = 0$ on I and $\varphi(x) = 0$ for $x \notin I$, we get that $\sum T(n)e^{inx} = 0$ for all x, thus $T = 0$. ■

Corollary 3.38. Let $K \in \mathcal{K}(\mathbb{T})$ and $\mathbb{M} = \mathcal{K}(\mathbb{T}) \setminus \mathcal{U}$. Then :

$$(K \in \mathbb{M}) \text{ if and only if } (\exists S \in PM : ||S||_\infty \leq 1, S \in PF, S \neq 0, \text{ and } K \text{ supports } S)$$

Proof. We note that the existence of some $S \in PM$ with $||S||_\infty \leq 1$, $S \in PF$ which is not trivially zero and such that it supported by K is equivalent to the existence of some $S \in PF$ such that $S \neq 0$ and that is supported by K.
Suppose that $K \in \mathbb{M}$, so that there is some $S \sim \sum c_n e^{inx}$ such that $S \neq 0$ and $\sum c_n e^{inx} = 0$ for $x \notin K$. Either $K \neq \mathbb{T}$ or $K = \mathbb{T}$. If $K \neq \mathbb{T}$, consider $S(n) = c_n$. By Proposition 3.14 one has that $c_n \to 0$ and thus, $S \in PF$. It follows by Theorem 3.36 that S is supported by K. Otherwise, if $K = \mathbb{T}$ we consider $S = \hat{\mu}$. In any case, we conclude that if $K \in \mathbb{M}$ then there is some $S \in PF$ such that $S \neq 0$ and that S is supported by K.
Conversely, suppose that there is some $S \in PF$ such that $S \neq 0$ and that is supported by K. By Theorem 3.36 it follows that $\sum S(n)e^{inx} = 0$ for $x \notin K$ and thus, $K \in \mathbb{M}$. ■

We can finally determine the complexity of $\mathcal{U}$:

Theorem 3.39. *The set of closed sets of uniqueness is coanalytic in $\mathcal{K}(\mathbb{T})$.*

Proof. We set again $\mathbb{M} = \mathcal{K}(\mathbb{T}) \setminus \mathcal{U}$ and we prove that $\mathbb{M} \in \mathbf{\Sigma^1_1}(\mathcal{K}(\mathbb{T}))$. We will consider $B_1(PM) = B_1(l^\infty) = B_1((l^1)^*)$ with the weak-$*$ topology. By Corollary 3.38, $\mathbb{M}$ is the projection of the following set :

$$P = \{(K,S) \in \mathcal{K}(\mathbb{T}) \times B_1(PM) : \lim_{|n|\to\infty} |S(n)| = 0, S \neq 0 \text{ and } K \text{ supports } S\}$$

Hence, it is enough to prove that P is Borel. We first note that for each n, $S \mapsto S(n) = \langle e^{-inx}, S\rangle$ is continuous and thus, $\lim_{|n|\to\infty} |S(n)| = 0$ is a Borel condition. Furthermore, $S \neq 0$ is also a Borel condition. It follows that it is enough to prove that the following set is closed :

[11] The assumptions of Theorem 3.22 hold since $S \in PF$ and since if $f \in C^p(\mathbb{T})$, then $|\hat{f}(n)| \leq \frac{M}{|n|^p}$.

$$Q = \{(K, S) \in \mathcal{K}(\mathbb{T}) \times B_1(PM) : \ K \text{ supports } S\}$$

Note that $B_1(PM)$ is compact (by Banach-Alaoglu Theorem) and metrizable (since PM is the dual of l^1, which is separable). Since $\mathcal{K}(\mathbb{T})$ is also metrizable, it is enough to prove that if $\{(K_i, S_i)\}_{i\in\omega} \subseteq Q$ is such that $K_i \to K$ and $S_i \to S$, then for interval I which is disjoint from K and any $\varphi \in C^\infty(\mathbb{T})$ supported by I, one has that $\langle \varphi, S\rangle = 0$. Let $supp(\varphi) \subseteq J \subseteq I$ with J closed and let $\mathcal{V} = \mathbb{T} \setminus J$, so that $K \subseteq \mathcal{V}$. By definition of Vietoris topology, $K_i \subseteq \mathcal{V}$ for all sufficiently large i. Hence, φ is supported by an interval disjoint from K_i and since $\langle \varphi, S_i\rangle \to \langle \varphi, S\rangle$, we conclude that $\langle \varphi, S\rangle = 0$. ■

Corollary 3.40. The set $\mathcal{U}$ is $\mathbf{\Pi^1_1}$-complete in $\mathcal{K}(\mathbb{T})$.

Proof. This follows from Theorems 3.2, 3.34 and 3.39. ■

3.4 Point spectrum

In this Section, we find the complexity of the point spectrum of a bounded operator acting on certain separable Banach spaces and other related sets. We will follow mainly [13] and [19]. We adapt some ideas present in [19] to determine the complexity of some sets which arise naturally whenever studying the point spectrum of operators.

We will denote the set of linear and bounded operators acting on a Banach space X by $\mathcal{L}(X)$. We recall that given some $T \in \mathcal{L}(X)$, the point spectrum of T is defined as follows :

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \ker(T - \lambda 1) \neq \{0\}\}$$

Theorem 3.41. *Let X be a separable and Banach space and $T \in \mathcal{L}(X)$. Then, $\sigma_p(T)$ is analytic.*

Proof. Let $W = \{x \in X : ||x|| = 1 \text{ and } \exists\lambda(x) \in \mathbb{C} : T(x) = \lambda(x)x\}$. Consider the map $\lambda : W \to \mathbb{C}$ such that $x \mapsto \lambda(x)$ and note that λ is well-defined. Moreover, $\lambda(W) = \sigma_p(T)$. Hence, it is enough to prove that λ is continuous and that W is closed.
Let $\{x_n\} \subseteq W$ be such that $x_n \to x \in X$. Clearly, $||x|| = 1$. Moreover, since $T(x_n) = \lambda(x_n)x_n$ and T is bounded, one has that :

$$|\lambda(x_n)| = \tfrac{||T(x_n)||}{||x_n||} \to \tfrac{||T(x)||}{||x||} < \infty$$

It follows that $\{\lambda(x_n)\}$ is bounded and thus, there is a convergent subsequence, say $\lambda(x_{n_k}) \to w$. Hence, $T(x_{n_k}) = \lambda(x_{n_k})x_{n_k} \to wx$. Thus, $x \in W$ and W is closed. Moreover, since $T(x) = wx$, one has that $w = \lambda(x)$. It follows that λ is continuous since X is a metric space and :

$$|\lambda(x_n)x - wx| \leq |\lambda(x_n)x - \lambda(x_{n_k})x_{n_k}| + |\lambda(x_{n_k})x_{n_k} - wx| \to 0$$

■

By Theorem 3.41, $\sigma_p(T)$ is at most in $\mathbf{\Sigma^1_1}$. A natural question arises :

If X is a separable Banach space and $T \in \mathcal{L}(X)$, is $\sigma_p(T)$ Borel ?

Kaufman proved in [13] that in general this is false, while P.Nimiec proved in [19] that if X is also reflexive, then $\sigma_p(T)$ is always Borel. In what follows, we will present both arguments and we finish this section with an adaptation of the arguments used in [19] to classify the complexity of other related sets. We start with Kaufman's result.

3.4.1 The point spectrum may not be Borel

In this subsection we present an example of a linear bounded operator acting on a separable Banach space which point spectrum is not a Borel set. This example strongly relies on a certain closed subspace of the space of Lipschitz and bounded maps from (any) separable and complete metric space to $\mathbb{C}$.

Definition 3.42. Let (X, d) be a metric space. A map $f : X \to \mathbb{C}$ is said to be Lipschitz if there is some $k \geq 0$ such that for all $x, y \in X$, then $|f(x) - f(y)| \leq kd(x, y)$. In this case, it is usual to call Lipschitz constant to the quantity :

$$\inf\{k \geq 0 : \forall x, y \in X : |f(x) - f(y)| \leq kd(x, y)\}$$

We will denote the set of all Lipschitz and bounded maps by $Lip(X, d)$.

Remark 3.43. Let (X, d) be a complete and separable metric space. We endow $Lip(X, d)$ with the following norm :

$$||f|| := ||f||_\infty + \sup\{\tfrac{|f(x)-f(y)|}{d(x,y)} : x \neq y\} := ||f||_\infty + ||f||_L$$

Then, $(Lip(X, d), ||.||)$ is a Banach space. Indeed, let $\{f_n\} \subseteq Lip(X, d)$ be a Cauchy sequence. In particular, it is a Cauchy sequence of continuous and bounded functions and thus, $f_n \to f$ (wrt $||.||_\infty$), for some f which is continuous and bounded. Moreover, for every $\epsilon > 0$ there is some N such that for any $n, m \geq N$ one has that $||f_n - f_m|| < \epsilon$. Thus, for any $x \neq y$ one has that :

$$\tfrac{|f_n(x)-f_m(x)-(f_n(y)-f_m(y))|}{d(x,y)} \leq ||f_n - f_m||_L \leq ||f_n - f_m|| < \epsilon$$

It follows that $f_n - f \in Lip(X, d)$ and thus, $f = f_n - (f_n - f) \in Lip(X, d)$ and it is clear that $||f_n - f|| \to 0$.

Henceforth, unless otherwise stated, we consider (X, d) to be a complete and separable metric space. It will be useful to define the following operation :

$$. : Lip(X, d) \times Lip(X, d)^* \to Lip(X, d)^*$$

$$(f.x^*)(g) := x^*(fg)$$

We let $E = \overline{\text{span}\{\epsilon_x\}}$, where each $\epsilon_x \in Lip(X,d)^*$ is such that $\epsilon_x(f) = f(x)$. It is easy to check that $||\epsilon_x - \epsilon_y|| \leq d(x,y)$ and thus, since (X,d) is assumed to be separable, so is E, i.e E is a separable Banach space.

Note that $(f.\epsilon_x)(g) = \epsilon_x(f.g) = f(x)g(x) = f(x)\epsilon_x(g)$, so that E is invariant under the operation previously defined. In other words :

$$f.\epsilon_x = f(x).\epsilon_x$$

Finally, for each $f \in Lip(X,d)$ we define $M_f \in \mathcal{L}(E)$ by :

$$M_f(x^*) := f.x^*$$

We note that for any $x \in X$, $M_f(\epsilon_x) = f.\epsilon_x = f(x).\epsilon_x$. It follows that :

$$f(X) \subseteq \sigma_p(M_f)$$

Our next goal is to prove the reverse inclusion :

Proposition 3.44. Let $f \in Lip(X,d)$. Then, $\sigma_p(M_f) = f(X)$.

Let us postpone for a while the proof of Proposition 3.44 and see how this implies that not every point spectrum of a bounded operator acting on a separable Banach space needs to be Borel :

Theorem 3.45. *Let $A \subseteq \mathbb{C}$ be a bounded and analytic subset. Then, A is the point spectrum of some bounded operator acting on a separable Banach space.*

Proof. Since A is analytic, let $f : X \to \mathbb{C}$ be a continuous map from some Polish space (X,d) such that $f(X) = A$. We define the following metric on X :

$$d'(x,y) := d(x,y) + |f(x) - f(y)|$$

We show that this is a complete metric on X under which $f \in Lip(X,d')$: Indeed, it is clear that f is bounded (since A is bounded) and that :

$$|f(x) - f(y)| \leq |f(x) - f(y)| + d(x,y) = d'(x,y)$$

Moreover, if $\{x_n\}$ is a Cauchy sequence in (X,d'), it certainly is as well in (X,d). Hence, $x_n \to x_0$ (wrt d) and since f is continuous, $f(x_n) \to f(x_0)$. It follows that $d'(x_n,x_0) \to 0$ and thus, (X,d') is complete. Furthermore, it is immediate that (X,d') is also separable.

Finally, by Proposition 3.44, $\sigma_p(T) = A = f(X)$ if we let $T = M_f \in \mathcal{L}(E)$. ∎

In order to prove the Proposition 3.44, we need the following two auxiliary results:

Proposition 3.46. Let $x^* \in E$. Then, for every $\epsilon > 0$ there is some finite subset $F \subset X$ and $A \in \mathbb{C}$ such that $||g.x^*|| \leq \epsilon||g|| + A\sum_{s\in F}|g(s)|$ for each $g \in Lip(X,d)$.

Proof. Let $x^* = \lambda_1\epsilon_{s_1} + \lambda_2\epsilon_{s_2}$ and take $F = \{s_1, s_2\}$. Note that $||g.x^*|| = \sup\{|g.x^*(f)| : ||f|| = 1\}$ and that if $||f|| \leq 1$, then $||f||_\infty \leq 1$. Thus :

$$|g.x^*(f)| \leq |\lambda_1||g(s_1)| + |\lambda_2||g(s_2)| \leq \max\{|\lambda_1|, |\lambda_2|\}(|g(s_1)| + |g(s_2)|)$$

It follows that the result holds for $x^* \in \text{span}(\epsilon_x)$.
Now consider a generic element $x^* \in E$ and $\epsilon > 0$. There is some finite set $F = \{s_1, ..., s_N\}$ such that $||x^* - \sum_{i=1}^{N}\lambda_i\epsilon_{s_i}|| < \epsilon$. Now, simply note that :

$$||g.x^*|| \leq ||g.(x^* - \sum_{i=1}^{N}\lambda_i\epsilon_{s_i})|| + ||g.\sum_{i=1}^{N}\lambda_i\epsilon_{s_i}||$$

Our result follows, since $\sum_{i=1}^{N}\lambda_i\epsilon_{s_i} \in \text{span}(\epsilon_x)$. ■

Proposition 3.47. Let $T \subseteq (X, d)$ and $h \in Lip(T, d)$. Then, one can extend h to an element $\hat{h} \in Lip(X, d)$ with the same Lipschitz constant.

Proof. Suppose that $|h(s) - h(t)| \leq Ld(s, t)$ for $s, t \in T$, with L the Lipschitz constant of h (as a map defined on $T \subset X$). We can set the following extension:

$$\hat{h}(s) := \inf\{h(t) + Ld(s, t), t \in T\}, \text{ for any } s \in X$$

The reader can verify the details in [30] (Theorem 1). ■

And now we can prove Proposition 3.44 :

Proof. (of Proposition 3.44) : It is enough to prove that $\sigma_p(M_f) \subseteq f(X)$. Suppose that $||x_0^*|| = 1$ is such that $M_f(x_0^*) = \lambda x_0^*$.

Step 1 : We construct elements $\{g_n\} \subseteq Lip(X, d)$ such that :

(i) $||g_1...g_n.x_0^*|| \geq \frac{1}{4} + \frac{1}{2^n}$, for $n \geq 1$.

(ii) The sets $S_n := \{s : g_n(s) \neq 0\}$ can be covered by a finite number of balls of radius $\frac{1}{n}$.

Let $n = 1$. We apply Proposition 3.46 with $\epsilon = 0.05$ and get a set $\{s_1, ..., s_N\}$. We define the following map :

$$g_1(s) = \begin{cases} 1 - \min\limits_{1\leq k\leq N} d(s, s_k), & \text{if } \min\limits_{1\leq k\leq N} d(s, s_k) < 1 \\ 0, & \text{otherwise} \end{cases}$$

Take $h(s) := 1 - g_1(s)$. Then, $||h.x_0^*|| \leq \epsilon||h|| + A\sum_k |h(s_k)| = \epsilon||h||$. Moreover, $||h||_\infty \leq 1$ and $||h||_L \leq 1$. To see that this is indeed the case, let $s \neq t$. Suppose

that $\min_k d(s, s_k) < 1$ and $\min_k d(t, s_k) > 1$, with k' such that $\min_k d(s, s_k)$ is attained. Then, $|h(t) - h(s)| \leq |d(t, s_{k'}) - h(s)| \leq d(s, t)$. All the remaining cases are similar and thus, we can conclude that $||h|| \leq 2$. It follows that $||h.x_0^*|| \leq 2\epsilon$ and thus, $||g_1.x_0^*|| \geq 0.9 > \frac{3}{4}$.
Now, suppose that all g_n have been constructed. Let $x^* = g_1 ... g_n x_0^*$ and $\epsilon = 4^{-2n} n^{-1}$ and define :

$$g_{n+1}(s) = \begin{cases} 1 - (n+1) \min\limits_k d(s, s_k), & \text{if } \min\limits_k d(s, s_k) < \frac{1}{n+1} \\ 0, & \text{otherwise} \end{cases}$$

Let $h = 1 - g_{n+1}$ and similarly as before we conclude that $||h.x^*|| < 2^{-n-1}$ whence one has that $||g_1 ... g_{n+1}.x_0^*|| = ||g_{n+1} x^*|| > 4^{-1} + 2^{-n} - 2^{-n-1} = 4^{-1} + 2^{-n-1}$. Furthermore, we note that the condition about S_n holds trivially, by definition of each g_n.

Step 2: Let $\mathcal{K} = \bigcap_n \overline{S_n}$. We note that $\mathcal{K}$ is compact, since it is complete and totally bounded. Indeed, $\mathcal{K}$ is closed, hence complete. Moreover, if $\mathcal{K}$ was not totally bounded, there is some $\epsilon > 0$ such that no finite collection of balls of radius ϵ covers $\mathcal{K}$. Then, there is some N such that $\frac{1}{N} < \epsilon$ and $\overline{S_N}$ cannot be covered by a finite number of balls of radius $\frac{1}{N}$ which would contradict the condition (ii) from the construction in Step 1.
Suppose, towards a contradiction, that $\lambda \notin f(\mathcal{K})$. For every $s \in \mathcal{K}$ one has that $f(s) \neq \lambda$ so there are open sets $\hat{\mathcal{V}}_s \in \mathcal{N}_{f(s)}$ and $\mathcal{U}_s \in \mathcal{N}_\lambda$ such that $\hat{\mathcal{V}}_s \cap \mathcal{U}_s = \emptyset$. By continuity, let $\mathcal{V}_s \in \mathcal{N}_s$ such that $f(\mathcal{V}_s) \subseteq \hat{\mathcal{V}}_s$ and since $\mathcal{K}$ is compact, let $\{\mathcal{V}_{s_i}\}_{i=1}^N$ cover $\mathcal{K}$. Let $\mathcal{V} = \bigcup\limits_{i=1}^N \mathcal{V}_{s_i}$ and $\lambda \in \mathcal{U} = \bigcap\limits_{i=1}^N \mathcal{U}_{s_i}$ so that $\mathcal{U} \cap \mathcal{V} = \emptyset$. We conclude that there is an open set $\mathcal{V} \supseteq \mathcal{K}$ and $\delta > 0$ such that $|f(s) - \lambda| > \delta$ for all $s \in \mathcal{V}$. Furthermore, note that there is some M such that $\bigcap\limits_n^N \overline{S_n} \subseteq \mathcal{V}$.
Then, $(f - \lambda)^{-1}$ is defined on $\mathcal{V}$, bounded by δ^{-1} and with Lipschitz constant $\delta^{-2}||f||$. By Proposition 3.47, let f_1 extend $(f - \lambda)^{-1}$ such that $f_1 \in Lip(X, d)$. For $n > M$ we have that :

$$g_1 ... g_n x_0^* = (f - \lambda) f_1 g_1 ... g_n x_0^* = f_1 g_1 ... g_n (f - \lambda) x_0^* = 0$$

Note that the last equality comes from our assumption that $M_f(x_0^*) = \lambda x_0^*$ and thus, $(f - \lambda) x_0^* = 0$. But this contradicts the condition (i) from the construction in Step 1. Therefore, we conclude that $\lambda \in f(\mathcal{K}) \subseteq f(X)$. ■

Remark 3.48. By Theorem 3.45 we answer negatively to the question of whether or not the point spectrum of a bounded operator acting on a separable Banach space is always Borel. However, if we restrict our attention to reflexive spaces, then the point spectrum is even a F_σ set ($\mathbf{\Sigma}_2^0$). This is a rather big *reduction* in complexity. In the next subsection, we shall establish this result.

3.4.2 The point spectrum is F_σ if X is reflexive

It is worth to remark that throughout this subsection, if we use terminology like weakly compact or weakly closed subsets of X - for a given Banach space X - we are referring to the weak topology on X, i.e the topology generated by the family of all linear functionals on X. Another central definition in the section is that of a reflexive space.

We start by recalling that a Banach space X is said to be **reflexive** if the inclusion $X \hookrightarrow X^{**}$ given by :

$$x \mapsto i(x)(f) := f(x), f \in X^*$$

is an isomorphism. We indicate some well-known properties and characterizations of reflexive Banach spaces:

(i) X is reflexive if and only if $\overline{B(0,1)}$ is weakly compact. In particular, if X is reflexive then it is weakly σ-compact.

(ii) X is reflexive if and only if X^* is reflexive.

(iii) If X is reflexive and $K \subseteq X$ is convex, bounded and closed, then K is weakly compact.

(iv) Any bounded sequence in a reflexive Banach space X has a weakly convergent subsequence.

(v) If X is reflexive and $C \subseteq X$ is a non-empty closed convex set, then there is some $c \in C$ (non necessarily unique) such that $||c|| = \inf_{x \in C} ||x||$.

Notation 3.49. Let $T \in \mathcal{L}(X)$, $K \subseteq X$ and $M \geq 0$. Then :

$$\Lambda_T(K, M) := \{\lambda \in \mathbb{C} : \ker(T - \lambda 1) \cap K \neq \emptyset \text{ and } |\lambda| \leq M\}$$

Proposition 3.50. Let $T \in \mathcal{L}(X)$, for X separable and Banach and let $K \subseteq X$ be weakly compact. Then, for any $M \geq 0$, the set $\Lambda_T(K, M)$ is compact.

Proof. If $0 \in K$, then $\Lambda_T(K, M) = \overline{B(0, M)}$, which is compact. Hence, we can assume that $0 \notin K$. We define :

$$W = \{x \in K : \exists \lambda(x) \in \mathbb{C} : T(x) = \lambda(x)x, |\lambda(x)| \leq M\}$$

It is clear that the map $\lambda : W \to \mathbb{C}$ such that $x \mapsto \lambda(x)$ is well-defined and that $\lambda(W) = \Lambda_T(K, M)$. Thus, it is enough to prove that W is weakly compact and that λ is weakly continuous, since the weak topology on $\mathbb{C}$ coincides with the usual euclidean topology. Moreover, since closed subsets of weakly compact spaces are still weakly compact, it will suffice to prove that W is closed in K. We note that since K is not necessarily weakly metrizable, we will use nets and sequential continuity in order to prove that λ is weakly continuous.

Let $\{x_\sigma\}_{\sigma\in\Sigma}$ be a net in W which is weakly convergent to some $x \in K$. It is enough to show that $x \in W$ and $\lim \lambda(x_\sigma) = \lambda(x)$. Since $\lambda(x_\sigma) \in \lambda(W)$ and $\lambda(W)$ is bounded, there is a convergent subnet $\lambda(x_\tau) \to w \in \mathbb{C}$. Since $T(x_\tau) = \lambda(x_\tau)x_\tau$, $(x_\tau, T(x_\tau))$ is weakly convergent to (x, wx). On the other hand, since T is bounded, the graph $\Gamma(T)$ is norm closed (and thus, as a linear subspace of $X\times X$, is also weakly closed[12]) so that $T(x) = wx$. Thus, $x \in W$ and $\lambda(x) = w$ and since $\lambda(W)$ is bounded, this also implies that $\lambda(x_\sigma) \to \lambda(x)$. ■

We recall the following classic result :

Theorem 3.51. *Every weakly compact subset of a separable Banach space X is weakly metrizable.*

Proposition 3.52. Let X be a separable Banach space and suppose that $X\setminus\{0\}$ is weakly σ-compact and that $T \in \mathcal{L}(X)$. Then, $\sigma_p(T)$ is a F_σ set.

Proof. Let $X \setminus \{0\} = \bigcup_n K_n$, with each K_n weakly compact. Note that $\lambda \in \sigma_p(T)$ if and only if $\lambda \in \Lambda_T(K_n, m)$ for some n and m. By Theorem 3.50, each $\Lambda_T(K_n, m)$ is compact and thus, $\sigma_p(T)$ is a F_σ set. ■

Proposition 3.53. Let $Y \subseteq X$ be a linear subspace of X. Then, $Y \setminus \{0\}$ is weakly σ-compact if and only if Y is weakly σ-compact and separable.

Proof. Suppose that Y is separable and that $Y = \bigcup_n K_n$, with each K_n weakly compact. By Theorem 3.51, each K_n is weakly metrizable. Since open sets of metrizable spaces are F_σ sets, it follows that $L_n = K_n \setminus \{0\}$ is a F_σ set in K_n with respect to the weak topology and thus, each L_n is weakly σ-compact. Since $Y \setminus \{0\} = \bigcup_n L_n$, it follows that $Y \setminus \{0\}$ is weakly σ-compact.
Conversely, suppose that $Y \setminus \{0\}$ is weakly σ-compact. It is immediate that Y is weakly σ-compact. Note that any $C \subseteq Y$ which is weakly closed, is then weakly G_δ : (here, $U_n \subseteq \mathbb{C}$ are open sets, h_n are linear functionals and we use the fact that closed sets in metric spaces are G_δ)

$$C = Y \setminus \bigcup_n h_n^{-1}(U_n) = \bigcap_n h_n^{-1}(\mathbb{C} \setminus U_n) = \bigcap_n h_n^{-1}(\bigcap_m V_m) = \bigcap_{n,m} h_n^{-1}(V_m)$$

In particular, $\{0\}$ is weakly G_δ and thus, it follows that there are linear functionals $\{f_n\}$ such that :

$$\{0\} = Y \cap \bigcap_n \ker(f_n)$$

We then define (for $\mathbb{C}^\omega$ endowed with the product topology) :

$$\Psi : Y \to \mathbb{C}^\omega$$
$$x \mapsto (f_n(x))_{n\in\omega}$$

Clearly, Ψ is injective and since each f_n is weakly continuous, Ψ is also weakly

[12] It follows from the Hahn-Banach Theorem that if X is a Banach space and $Y \subset X$ is a linear subspace, then the norm closure of Y coincides with its weak closure.

continuous. Thus, Ψ is an embedding when restricted to any weakly compact subset of Y. Let $Y = \bigcup_n K_n$, with each K_n weakly compact. It follows that $\Psi|_{K_n}$ is an embedding and thus, each K_n is weakly separable and so is Y. Hence, there is some countable and weakly dense subset $D \subseteq Y$. Take $B = \overline{\text{span}(D)}$. It remains to note that B is separable, weakly closed and that $Y \subseteq B$. ■

Corollary 3.54. Let X be a reflexive and separable Banach space and $T \in \mathcal{L}(X)$. Then, $\sigma_p(T)$ is a F_σ set.

Proof. By Proposition 3.53, $X \setminus \{0\}$ is weakly σ-compact and by Proposition 3.52, $\sigma_p(T)$ is then a F_σ set. ■

3.4.3 An analytic set of $\mathcal{F}(X)$ which is not Borel

We finish this section with an adaptation of the arguments in [19] and we characterize some sets that arise naturally in the setting of point spectrum. In particular, we show that if X is a separable and reflexive Banach space and $T \in \mathcal{L}(X)$, then $\{F \in \mathcal{F}(X) : F \text{ is uncountable and } F \neq \ker(T - \lambda 1)\}$ is analytic and not Borel. This contrasts with what happens at the level of the point spectrum (Corollary 3.54).

Definition 3.55. Let X be a Banach space and $T \in \mathcal{L}(X)$. We define a map $\Gamma : \mathbb{C} \to \mathcal{F}(X)$ such that :

$$\lambda \mapsto \ker(T - \lambda 1)$$

Notation 3.56. Let X be a Banach space and let d be its norm induced metric. Let $r, p > 0$, $x \in X$ and $T \in \mathcal{L}(X)$. We define :

$$C^x_{r,p} = \{y \in X : r \leq ||y||, d(x,y) \leq p \text{ and } \exists \lambda(y) \in \mathbb{C} : T(y) = \lambda(y)y\}$$

Since $0 \notin C^x_{r,p}$, the map $\varphi^x_{r,p} : C^x_{r,p} \to \mathbb{C}$ given by $y \mapsto \lambda(y)$ is well-defined.

Notation 3.57. Let X be a Banach space, $T \in \mathcal{L}(X)$, $x \in X$ and $p \geq 0$. We define the following sets :

$$A^p_x = \{\lambda \in \mathbb{C} : d(x, \ker(T - \lambda 1)) \leq p\} \text{ and } \hat{A}^p_x = A^p_x \cap \sigma_p(T)$$

Theorem 3.58. *Let X be a Banach space and $T \in \mathcal{L}(X)$ such that $\sigma_p(T)$ is Borel. Then, every A^p_x is analytic.*

Proof. **Step 1**: First, we prove that $\varphi^x_{r,p}(C^x_{r,p}) \subseteq \mathbb{C}$ is analytic. It is enough to prove that $C^x_{r,p} \subseteq X$ is closed and that $\varphi^x_{r,p}$ is continuous. Let $\{y_n\} \subseteq C^x_{r,p}$ such that $y_n \to y$. Clearly, $||y|| \geq r$. Moreover, and since T is bounded :

$$|\lambda(y_n)| = \tfrac{||T(y_n)||}{||y_n||} \to \tfrac{||T(y)||}{||y||} < \infty$$

Thus, $\{\lambda(y_n)\}$ is bounded and so we can consider a convergent subsequence $\lambda(y_{n_k}) \to \lambda$. It follows that $T(y_{n_k}) = \lambda(y_{n_k})y_{n_k} \to \lambda y$. Hence, $T(y) = \lambda y$.

Furthermore, it is clear that $d(x,y) \leq p$ and thus, $C^x_{r,p}$ is closed. We finally note that we also proved that $\varphi^x_{r,p}$ is sequentially continuous, hence continuous.

Step 2: We prove that for each $p > 0$, the sets $\hat{A}^p_x$ are analytic. We divide the analysis in two cases :
(i) Suppose that $d(x,0) \leq p$. Then, $\hat{A}^p_x = \sigma_p(T)$ which is Borel by assumption (even analytic by Theorem 3.41).
(ii) Suppose that $d(x,0) > p$. By Step 1, it is enough to prove that for each $r_m = \frac{1}{m}$, the following holds :

$$\hat{A}^p_x = \bigcup_m \bigcap_n \varphi^x_{r_m,p+\frac{1}{n}}(C^x_{r_m,p+\frac{1}{n}})$$

Indeed, let $\lambda \in \hat{A}^p_x$. Since $d(x,0) > p$, for every n one can choose some $y_n \neq 0$ such that $y_n \in \ker(T - \lambda 1)$ and $d(x, y_n) \leq p + \frac{1}{n}$. One can assume without loss of generality that there is some $\delta > 0$ such that $||y_n|| \geq \delta$, for all n. Indeed, towards a contradcition, suppose that for every $\epsilon > 0$ there is some N such that for all $k \geq N$ then $d(y_k, 0) < \epsilon$. Then, $y_n \to 0$ which implies that $d(x,0) \leq p$. Hence, there is some $\delta > 0$ such that for all n, there is some $k \geq n$ such that $d(y_k, 0) \geq \delta$. Thus, we can assume without loss of generality that that $||y_n|| \geq \delta$. If $\delta \geq 1$, then certainly $\lambda \in \varphi^x_{r_1,p+\frac{1}{n}}(C^x_{r_1,p+\frac{1}{n}})$ for all n. Otherwise, let m be such that $\frac{1}{m} \leq \delta$. In this case, it is clear that $\lambda \in \varphi^x_{r_m,p+\frac{1}{n}}(C^x_{r_m,p+\frac{1}{n}})$ for all n.
Conversely, let $\lambda \in \bigcap_n \varphi^x_{r,p+\frac{1}{n}}(C^x_{r,p+\frac{1}{n}})$ for some $r > 0$. Then, for all n there is some $y_n \in \ker(T - \lambda 1)$ such that $||y_n|| \geq r > 0$ and $d(x, y_n) \leq p + \frac{1}{n}$. Thus, $\lambda \in \sigma_p(T)$. If $d(x, \ker(T - \lambda 1)) > p$, then there is some l such that for all $y \in \ker(T - \lambda 1)$, $d(x,y) \geq p + \frac{1}{l}$, which contradicts our assumption. Hence, $\lambda \in \hat{A}^p_x$.

Step 3: (i) Let $p > 0$. By Step 2, $\hat{A}^p_x$ is analytic and thus, it is enough to prove that $A^p_x \cap \neg\sigma_p(T)$ is analytic. However, either $A^p_x \cap \neg\sigma_p(T) = \neg\sigma_p(T)$ or $A^p_x \cap \neg\sigma_p(T) = \emptyset$. By assumption, $\sigma_p(T)$ is Borel, so we are done.
(ii) If $p = 0$, then either $A^0_0 = \mathbb{C}$ or A^p_x has at most one element, for $x \neq 0$. In either case, A^0_x is analytic. ■

Theorem 3.59. *Let X be a separable and reflexive Banach space, with $T \in \mathcal{L}(X)$. Then, A^p_x is Borel.*

Proof. For any $x \in X$, let $K^r_x = \overline{B(x,r)}$. Since X is reflexive, it follows by Theorem 3.50 that each $\Lambda_T(K^r_x, M)$ is compact and thus, it is enough to prove that :

$$A^p_x = \bigcup_M \bigcap_n \Lambda_T(K^{p+\frac{1}{n}}_x, M)$$

Let $\lambda \in A^p_x$ such that $|\lambda| \leq M$. Then, $d(x, \ker(T - \lambda 1)) \leq p$ and thus, for any n, there is some $y_n \in \ker(T - \lambda 1) \cap K^{p+\frac{1}{n}}_x$.
Conversely, suppose that for some M, $\lambda \in \Lambda_T(K^{p+\frac{1}{n}}_x, M)$ for every n. If $d(x, \ker(T - \lambda 1)) > p$, then there is some m such that for every $y \in \ker(T - \lambda 1)$

one has that $d(x,y) \geq p + \frac{1}{m}$. It follows that $\lambda \notin \Lambda_T(K_x^{p+\frac{1}{m+1}}, M)$, which contradicts our assumption. $\blacksquare$

Suppose that (X, Σ) is a measurable space and that Y is a second countable Hausdorff space, which topology is generated by some subbasis $\mathcal{S}$. Let $\mathcal{B}(\mathcal{S})$ be the Borel algebra generated by its topology. Then, it is a well-known fact that $f : (X, \Sigma) \to (Y, \mathcal{B}(\mathcal{S}))$ is measurable if and only if $f^{-1}(S) \in \Sigma$, for every $S \in \mathcal{S}$ (cf [2]).

Heceforth, and unless otherwise stated, when we consider the map Γ, we consider $\mathcal{F}(X)$ endowed with the Wijsman topology. We note that since $\sigma_p(T) = \Gamma^{-1}(\mathcal{F}(X) \setminus \{0\})$, if Γ is measurable, then $\sigma_p(T)$ is Borel.

Theorem 3.60. *Let X be a separable and reflexive space. Then, Γ is a measurable map.*

Proof. We recall that the Wijsman topology on $\mathcal{F}(X)$ is the weak topology generated by the family of maps $\{\varphi_x\}_{x \in X}$, with $\varphi_x(A) = d(x, A)$ (see section 4.4). In order to prove that Γ is a measurable map, by Theorem 4.85, it is enough to show that $\Gamma^{-1}(\varphi_x^{-1}((p,q)))$ is Borel for any $x \in X$ and $p, q \in \mathbb{Q}_0^+$. It suffices that :

$$B_x^p = \{\lambda \in \mathbb{C} : d(x, \ker(T - \lambda 1)) > p\} \text{ and } \{\lambda \in \mathbb{C} : d(x, \ker(T - \lambda 1)) < q\}$$

are Borel. Note that $B_x^p = \neg A_p^x$ and thus, by Theorem 3.59, is Borel. Moreover, note that $C_x^p = \bigcup_n A_x^{q-\frac{1}{n}}$ and thus, is Borel. $\blacksquare$

Remark 3.61. Let X be a separable and Banach space and $T \in \mathcal{L}(X)$. Then, if $\sigma_p(T)$ is countable, the map Γ is measurable. In particular, if T is power bounded then Γ is measurable since it is well-known that $\{n_k\}$ with $n_k = k$ is a Jamison sequence (cf [21] and [20]).

We finish this Section with a proof that, for a fixed $T \in \mathcal{L}(X)$, acting on a separable reflexive Banach space, the set of uncountable closed sets $F \in \mathcal{F}(X)$ such that avoid to be of the form $\ker(T - \lambda 1)$, is analytic but not Borel (with respect to the Effros Borel space). In order to do so, we will use the following well-known result :

Proposition 3.62. Let X and Y be standard Borel spaces and $f : X \to Y$ be a Borel map. If $A \subseteq X$ is Borel and $f|_A$ is injective, then $f(A)$ is Borel.

Proof. The reader can check [1]. $\blacksquare$

Corollary 3.63. If X is a separable and reflexive Banach space and $T \in \mathcal{L}(X)$, then the following set is Borel :

$$\{\ker(T - \lambda 1) : d(x, \ker(T - \lambda 1)) \leq p\}$$

Proof. It follows immediately by Theorem 3.59 and 3.60 and Proposition 3.62. ■

Theorem 3.64. *Let X be a separable and reflexive Banach space, with $T \in \mathcal{L}(X)$. Let $\mathcal{R} = \{F \in \mathcal{F}(X) : F$ is uncountable and $F \neq \ker(T - \lambda 1)\}$. Then, $\mathcal{R}$ is analytic but not Borel (wrt Effros Borel structure).*

Proof. By Theorem 3.54, $\sigma_p(T)$ is Borel and by Theorem 3.60, Γ is a measurable map (wrt the Borel algebra generated by the Wijsman topology). Since $\Gamma|_{\sigma_p(T)}$ is injective, it follows by Proposition 3.62 and Theorem 4.85 that $\Gamma(\sigma_p(T))$ is an element of the Effros Borel space of $\mathcal{F}(X) \setminus \{\emptyset\}$. By Theorem 3.3, it follows that $\mathcal{R} = \{F \in \mathcal{F}(X) : \text{F is uncountable}\} \setminus \Gamma(\sigma_p(T))$ is analytic. Moreover, if $\mathcal{R}$ is Borel, then $\{F \in \mathcal{F}(X) : \text{F is uncountable}\}$ would not be $\mathbf{\Sigma}^1_1$-hard, contradicting Theorem 3.3. ■

4 Appendix

4.1 Ordinal numbers and Independence

In this section, we provide a short introduction to ordinal (and cardinal) numbers and, for the sake of completeness, some extended comments on models and independence results.

Throughout the subsections of section 4.1 we work, unless otherwise stated, within ZFC. Moreover, for the sake of readibility, the majority of standard notions (such as products, power sets, functions, orderings...) are assumed to be objects which are *available for free* and so, we do not define them from the axioms.

This section of the appendix should not be seen as a rigourous approach to Set Theory or Mathematical Logic, but rather as a point of reference for the reader who is not familiar with some of the concepts which are used throughout the thesis. As a consequence, we will ommit most of the proofs.

4.1.1 Ordinal numbers

We start with the central notion of well-ordering :

Definition 4.1. A total order $<$ on a set X is said to be a well-ordering if every non-empty subset of X has a least element with respect to $<$. In this case, $(X, <)$ is said to be a well-ordered set.

The concepts of (order) isomorphism and initial segments are also essential for our understanding of well-orderings. Furthermore, Theorem 4.5 relates both concepts in a fundamental way.

Definition 4.2. Let $(P, <)$ and $(Q, <')$ be partially ordered sets and $f : P \to Q$. Then, f is said to be order-preserving if :

$$\forall x, y \in P : x < y \Rightarrow f(x) <' f(y)$$

If $f : P \to Q$ is injective and f, f^{-1} are order-preserving, then f is said to be an isomorphism and $(P, <)$ and $(Q, <')$ are said to be isomorphic.

Definition 4.3. Let $(W, <)$ be a well-ordered set. Then, any set of the form $W(y) = \{x \in W : x < y\}$ (for some $y \in W$) is said to be an initial segment of W.

Proposition 4.4. No well-ordered set is isomorphic to an initial segment of itself.

Proof. Suppose that W is a well-ordered set and that there is an isomorphism $f : W \to W(y)$ for some $y \in W$. Then, the set $M = \{x \in W : f(x) \neq x\}$ is non-empty and thus, let m be its minimal element. If $f(m) < m$, then $f(m)$ contradicts the minimality of m. If, on the other hand $f(m) > m$, it follows that $m \in W(y)$. Let $a \in W$ such that $f(a) = m$, so that $a \in M$ and thus, it contradicts the minimality of m. ■

Theorem 4.5. *If W_1 and W_2 are two well-ordered sets, then exactly one of the following holds :*

(i) W_1 and W_2 are isomorphic.

(ii) W_1 is isomorphic to an initial segment of W_2.

(iii) W_2 is isomorphic to an initial segment of W_1.

Proof. The reader can find a proof in [7] (Theorem 2.8, p. 18). ■

We define ordinal numbers to be sets with certain properties :

Definition 4.6. A set X is transitive iff $y \in X \Rightarrow y \subseteq X$.

For instance, $\emptyset$ and $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ are transitive, while $\{\{\emptyset\}\}$ is not.

Definition 4.7. An **ordinal number** (or simply ordinal) is a transitive set which is well-ordered by $\in$. The class of all ordinals in denoted by **Ord**.

Henceforth, for ordinals α, β, we define that :

$$\alpha < \beta \text{ if and only if } \alpha \in \beta$$

The following result shows that $<$ is a total order of the class of ordinals :

Theorem 4.8. *The following statements are true :*

(i) $\emptyset$ is an ordinal.

(ii) If α is an ordinal and $\beta \in \alpha$, then β is an ordinal.

(iii) If α, β are distinct ordinals and $\alpha \subseteq \beta$, then $\alpha \in \beta$.

(iv) If α and β are ordinal, then either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

Proof. The reader can find a proof in [7] (Theorem 2.11, p. 19). ∎

In order to get a better intuition on ordinals, let α be an ordinal. If $\gamma < \alpha$, then it follows by definition that $\gamma \in \alpha$ and, by Theorem 4.8, we conclude that $\gamma \in \mathbf{Ord}$. Hence, we have that :

$$\alpha = \{\beta \in \mathbf{Ord} : \beta < \alpha\}$$

It is also worth to note that **Ord** is not a set, but rather a *proper* class :

$$\nexists Y \forall X : X \in \mathbf{Ord} \Rightarrow X \in Y$$

Indeed, suppose towards a contradiction that **Ord** is a set. By the previous comment, if α is an ordinal, then $\alpha = \{\beta \in \mathbf{Ord} : \beta < \alpha\}$ and thus, **Ord** would be a transitive set. Furthermore, **Ord** is well-ordered by $<$: Given any non-empty set C of ordinals, there exists some $x \in C$ such that for any $y \in C$, either $x \in y$ or $x = y$ - we simply take $x = \bigcap C$. Therefore, if **Ord** is a set, it would be an ordinal and we would have that $\mathbf{Ord} \in \mathbf{Ord}$, which contradicts the Axiom of Foundation.

In the previous remark about **Ord** being a class, we implicitly used the fact that $\bigcap C$ is still and ordinal. Indeed, one can prove that if C is any non-empty *class* of ordinals, then $\bigcap C \in \mathbf{Ord}$ and if X is any non-empty set of ordinals, then $\bigcup X \in \mathbf{Ord}$. This motivates the following definition :

Definition 4.9. Let X be a non-empty set of ordinals and C a non-empty class of ordinals. Then, we denote $\inf C = \bigcap C$ and $\sup X = \bigcup X$.

Now, suppose that there are ordinals $\alpha \neq \beta$ which are isomorphic. It follows by Theorem 4.8 that either $\alpha \in \beta$ or $\beta \in \alpha$ which contradicts Proposition 4.4. Thus, if α and β are isomorphic, it follows that $\alpha = \beta$. Indeed, this antecipates the next theorem which is a consequence of Theorem 4.5 and our definition of ordinal, providing additional intuition on ordinal numbers : in a sense, they fully *capture* the order-type of *all* well ordered sets :

Theorem 4.10. *Every well-ordered set is isomorphic to an unique ordinal.*

Proof. The reader can find a proof in [7] (Theorem 2.12, p. 20). ∎

Now, we define successor and limit ordinals. The least limit ordinal will coincide with our definition of natural numbers. Before doing so, note that if α is an ordinal, then $\alpha \cup \{\alpha\}$ is still an ordinal.

Definition 4.11. Let α be an ordinal. Then, $\alpha + 1 := \alpha \cup \{\alpha\}$ is called the successor of α.

Sometimes, we denote $\alpha + 1$ by $S(\alpha)$. It is immediate that $\alpha < S(\alpha)$ and that for every $\beta \in \mathbf{Ord}$, if $\beta < S(\alpha)$, then $\beta \leq \alpha$.

Definition 4.12. An ordinal α is said to be a **successor ordinal** if there is some ordinal β such that $\beta + 1 = \alpha$. Otherwise, α is said to be a **limit ordinal**.

One can finally define the set of natural numbers :

Definition 4.13. An ordinal α is a **natural number** if and only if :

$$\forall \beta \leq \alpha, \text{ either } \beta = 0 \text{ or } \beta \text{ is a successor ordinal}$$

The set of natural numbers will be denoted by ω.

Using the usual notation for natural numbers : $0 := \emptyset$, $1 := \{0\}$, $2 := \{0, 1\}$, $3 := \{0, 1, 2\}$ and so on. Furthermore, such set ω exists by the Axiom of Infinity.

Remark 4.14. One can easily show that ω is the least limit ordinal which is not zero. It is reasonable to ask whether or not our definition of ω *really captures the essence of the natural numbers*. In this sense, it is worth to mention that ω satisfy the Peano Axioms : (c.f [6], Theorem 7.16, p. 19)

(i) $0 \in \omega$

(ii) $\forall n \in \omega : S(n) \in \omega$

(iii) $\forall n, m \in \omega : n \neq m \Rightarrow S(n) \neq S(m)$

(iv) $\forall X \subseteq \omega : (0 \in X)$ and $(\forall n \in X : S(n) \in X)$, then $X = \omega$

The fourth Peano Axiom is essentially what we usually call principle of induction. However, the only property of natural numbers which is needed in order to use induction is its well-ordering. Thus, it should not be surprising that one can extend the principle of induction for other well-ordered sets.

Theorem 4.15. ***(Transfinite induction)*** *Let C be a class of ordinals and assume that :*

(i) $0 \in C$

(ii) If $\alpha \in C$, then $\alpha + 1 \in C$

(iii) If α is a non zero limit ordinal and $\beta \in C$ for all $\beta < \alpha$, then $\alpha \in C$.

Then, $C = \mathbf{Ord}$.

Proof. Suppose, towards contradiction, that there is some ordinal which is not in C. Then, there is a minimal such ordinal - α. By (i), $\alpha \neq 0$. If α is a successor ordinal, then $\alpha = \beta + 1$ and it follows by (ii) that β contradicts the minimality of α. Similarly, for the case with α a limit ordinal, by using (iii). ∎

It is often very useful to define recursively an object. It is now to be expected that one can define objects also by *transfinite methods* and this is indeed the content of the next theorem. In order to state this result, we introduce some terminology which shall be used later :

Definition 4.16. A transfinite α-sequence (of length α) is a function whose domain is an ordinal, say $\langle a_\xi : \xi < \alpha \rangle$.

Theorem 4.17. ***(Transfinite recursion)*** *Let G be a function defined on* **Ord**. *Then, there is an unique function F on* **Ord** *such that for every $\alpha \in$* **Ord** *one has that $F(\alpha) = G(F|_\alpha)$.*
In particular, let X be a set and θ be an ordinal. Then, for every function G on the set of transfinite sequences in X of length $< \theta$ such that range(G)$\subseteq X$, there is an unique θ-sequence $\{a_\alpha : \alpha < \theta\}$ in X such that $a_\alpha = G(\{a_\xi : \xi < \alpha\})$ for every $\alpha < \theta$.

Proof. The reader can find a proof in [7] (Theorem 2.15, p.22). ■

As an application of Theorem 4.17, we show that Bernstein sets exist :

Definition 4.18. A set $B \subseteq \mathbb{R}$ is called a Bernstein set if it intersects every uncountable closed set of $\mathbb{R}$ but it does not contain any such closed set.

Proposition 4.19. Bernstein sets exist.

Proof. By the Axiom of Choice, one can consider a well-ordering $<$ on $\mathbb{R}$. We let α be the least ordinal which is in bijection with $\mathbb{R}$. Moreover, it follows by Proposition 1.5 that there is a bijection between the set of uncountable closed subsets of $\mathbb{R}$ and $\mathbb{R}$. Furthermore, it follows from Theorems 4.63 and 4.64 that each uncountable closed subset $C \subseteq \mathbb{R}$ is such that $|C| = |\mathbb{R}|$. We can then index all the uncountable closed subsets of $\mathbb{R}$ by $\{C_\beta\}_{\beta<\alpha}$.
Let x_0, y_0 be the first two elements (wrt $<$) in C_0. We will construct recursively two families $\{x_\beta\}_{\beta<\alpha}$ and $\{y_\beta\}_{\beta<\alpha}$ of points : We pick x_1, y_1 to be the first two elements in $C_1 \setminus \{x_0, y_0\}$. Now, for a fixed $\gamma < \alpha$ we assume that we have picked elements $x_\beta, y_\beta \in C_\beta$ for every $\beta < \gamma$. Since α is minimal among all ordinals in bijection with $\mathbb{R}$, one can pick the first two elements in $C_\gamma \setminus \bigcup_{i<\gamma}\{x_i, y_i\}$.
We define $B = \{x_\beta\}_{\beta<\alpha}$ and it follows from construction that B intersects each C_β. Moreover, B does not contain any C_β since $y_\beta \notin B$. ■

As another application of transfinite recursion, we now define the operations of ordinal arithmetic. But first, we introduce some useful notation : let α be a non zero limit ordinal and $\{\gamma_\xi : \xi < \alpha\}$ be a nondecreasing sequence of ordinals. Then, we define $\lim_{\xi\to\alpha} \gamma_\xi := \sup\{\gamma_\xi : \xi < \alpha\}$.

Definition 4.20. (**Addition**) For every $\alpha \in$ **Ord** we define :

(i) $\alpha + 0 = \alpha$

(ii) $\alpha + (\beta + 1) = (\alpha + \beta) + 1$, for every $\beta \in$ **Ord**

(iii) $\alpha + \beta = \lim_{\xi \to \beta}(\alpha + \xi)$, for every non zero limit ordinal β

Definition 4.21. (**Multiplication**) For every $\alpha \in \mathbf{Ord}$ we define :

(i) $\alpha.0 = 0$

(ii) $\alpha.(\beta + 1) = \alpha.\beta + \alpha$, for every $\beta \in \mathbf{Ord}$

(iii) $\alpha.\beta = \lim_{\xi \to \beta} \alpha.\xi$, for every non zero limit ordinal β

Definition 4.22. (**Exponentiation**) For every $\alpha \in \mathbf{Ord}$ we define :

(i) $\alpha^0 = 1$

(ii) $\alpha^{\beta+1} = \alpha^\beta.\alpha$, for every $\beta \in \mathbf{Ord}$

(iii) $\alpha^\beta = \lim_{\xi \to \beta} \alpha^\xi$, for every non zero limit ordinal β

We finish this section with the Cantor's normal form Theorem :

Theorem 4.23. *Every ordinal $\alpha > 0$ can be uniquely represented in the form :*

$$\alpha = \omega^{\beta_1}.k_1 + ... + \omega^{\beta_n}.k_n$$

where $n \geq 1$, $\alpha \geq \beta_1 > ... > \beta_n$ and k_i are non zero natural numbers.

Proof. The reader can find a proof in [7] (Theorem 2.26, p. 24). ■

4.1.2 Cardinal numbers

We start with the notion of *equinumerous* sets :

Definition 4.24. Let X and Y be any sets. Then :

(i) $|X| \leq |Y|$ if there is an injective map from A to B

(ii) $|X| = |Y|$ if there is a bijection between X and Y

(iii) $|X| < |Y|$ if $|X| \leq |Y|$ and $|X| \neq |Y|$

If $|X| = |Y|$, we say that X and Y have the same cardinality and if $|X| < |Y|$, we say that X has smaller cardinality than Y. Cantor's Theorem shows that the concept of cardinality is not trivial (even among non-finite sets) :

Theorem 4.25. *For any set X, one has that $|X| < |\mathcal{P}(X)|$.*

Proof. It is clear that $|X| \leq |\mathcal{P}(X)|$. In order to prove that $|X| < |\mathcal{P}(X)|$, we note that for any map $f : X \to \mathcal{P}(X)$, the set $Y = \{x \in X : x \notin f(x)\}$ is not in the range of f. ■

The next theorem is known as the Cantor-Bernstein theorem and it is provable in ZF. It shows that $<$ is a partial ordering :

Theorem 4.26. *If* $|X| < |Y|$ *and* $|Y| < |X|$, *then* $|X| = |Y|$.

Proof. The reader can find a proof in [7] (Theorem 3.2, p. 28). ■

Remark 4.27. Recall that the Axiom of Choice is equivalent to the statement that every set can be well-ordered. Hence, it follows that within ZFC $<$ is a total order. However, it is worth to note that it can be shown that the Axiom of Choice is **equivalent** to the following tricothomy :

For any sets X and Y, either $|X| \leq |Y|$, $|X| \geq |Y|$ or $|X| = |Y|$

We can now define a cardinal number :

Definition 4.28. An ordinal number α is a **cardinal number** (or simply cardinal) if $|\alpha| \neq |\beta|$ for all $\beta \leq \alpha$.

Definition 4.29. Let W be a well-ordered set. Then, its **cardinality** is :

$$|W| = \min\{\alpha \in \mathbf{Ord} : |\alpha| = |W|\}$$

Remark 4.30. Within ZFC, the cardinality of any set can be defined as in Definition 4.29. If we work within ZF, one can still assign a cardinality to every set via Scott's trick.

We can now formally define what we mean by finite and infinite sets :

Definition 4.31. A set X is said to be finite if $|X| = n$ for some $n \in \omega$. Otherwise, we say that X is infinite.

We note that ω is the least infinite cardinal and that each infinite cardinal is necessarily a limit ordinal.

Remark 4.32. A set X is said to be Dedekind infinite if there is some injective map $f : X \to X$ which is non surjective. Otherwise, X is said to be Dedekind finite. If one works within ZFC, the notions of finite (infinite) set and Dedekind finite (infinite) set coincide. However, there are models of ZF where amorphous sets exist. An amorphous set is an infinite set which is not a disjoint union of two infinite subsets. Hence, there are models of ZF with infinite sets which are Dedekind finite.

Definition 4.33. A set X is said to be countable if $|X| \leq \omega$. Otherwise, X is said to be uncountable.

A particular ordinal which is quite important (also in this thesis) is the following :

Definition 4.34. The smallest ordinal which is uncountable as a set is denoted by ω_1.

It follows from Theorem 4.25 that there are uncountable sets. Assuming the Axiom of Choice, each uncountable set is assigned with a cardinal so that we can define the class of uncountable ordinals. The infimum of this class, coincides with ω_1. Alternatively, one can see ω_1 as the supremum (or union) of all countable ordinals. If we work within ZF, it is still possible to define ω_1 via the Hartog's number :

Let X be any set. We define :

$$\alpha_X = \{\beta \in \mathbf{Ord} : \text{ there exists an injective map } i : \beta \to X\}$$

One can prove in ZF that α_X is an ordinal number. Furthermore, α_X is the least ordinal α such that there is no injective map $i : \alpha \to X$. Hence, we can define $\omega_1 = \alpha_\omega$.

Using the same method, for every cardinal κ there is the least cardinal number which is greater than κ. It is usual to denote it by κ^+. It is also standard to use the *aleph* notation. For $\alpha \in \mathbf{Ord}$, we define :

$$\aleph_0 := \omega$$

$$\aleph_{\alpha+1} := \aleph_\alpha^+ := \omega_{\alpha+1}$$

$$\text{If } \alpha \text{ is a limit ordinal, } \aleph_\alpha := \sup\{\omega_\beta : \beta < \alpha\}$$

It is usual to use $\aleph_\alpha$ to refer to the cardinal number and ω_α to denote its order type.

We now define the operations of cardinal arithmetic :

Definition 4.35. Let κ and λ be cardinals. Then :

(i) $\kappa + \lambda := |X \cup Y|$, where $|X| = \kappa$, $|Y| = \lambda$ and $X \cap Y = \emptyset$

(ii) $\kappa.\lambda := |X \times Y|$, where $|X| = \kappa$ and $|Y| = \lambda$

(iii) $\kappa^\lambda := |X^Y|$, where $|X| = \kappa$ and $|Y| = \lambda$

In what follows, we state several results about cardinal arithmetic which are often used.

Theorem 4.36. *Let κ and λ be infinite cardinals. Then :*

(i) $\kappa + \lambda = \kappa.\lambda = \max\{\kappa, \lambda\}$

(ii) $|\kappa^{<\omega}| = \kappa$

Proof. The reader can find a proof in [6] (Theorem 10.12 and Corollary 10.13, p. 29). ■

In particular, if X is an infinite set, then X and $X \times X$ have the same cardinality. It is a theorem due to Tarski that indeed this statement is **equivalent** to the Axiom of Choice. Hence, the Axiom of Choice is needed in order for Theorem 4.36 to hold.

Theorem 4.37. *Let κ be a cardinal such that $\kappa \geq \omega$ and, for all $\alpha < \kappa$ let X_α be a set such that $|X_\alpha| \leq \kappa$. Then, $|\bigcup_{\alpha<\kappa} X_\alpha| \leq \kappa$.*

Proof. The interested reader can find a proof in [6] (Lemma 10.21, p. 30). ■

In particular, a countable union of countable sets is still countable. However, it is worth to note that there are models of ZF where this fails. For instance, there are models of ZF where $\mathcal{P}(\omega)$ and ω_1 are actually countable unions of countable sets. Hence, the Axiom of Choice is needed in order for Theorem 4.37 to hold.

Now, consider an indexed family of cardinals $\{\kappa_i\}_{i\in I}$. We define :

$$\sum_{i\in I} \kappa_i := |\sqcup X_i|, \text{ with each } |X_i| = \kappa$$

$$\prod_{i\in I} \kappa_i := |\prod_{i\in I} X_i|, \text{ with each } |X_i| = \kappa_i$$

Theorem 4.38. *Let λ be an infinite cardinal and κ_i be non zero cardinals for each $i < \lambda$. Then :*

(i) $\sum_{i<\lambda} \kappa_i = \lambda.(\sup_{i<\lambda} \kappa_i)$

(ii) If $\langle \kappa_i : i < \lambda\rangle$ is non decreasing, then $\prod_{i<\lambda} \kappa_i = \lambda.(\sup_i \kappa_i)$

Proof. The reader can find a proof in [7] (Lemma 5.8, p.52 and Lemma 5.9, p. 54). ■

The next theorem is known as Konig's Theorem and relates in a fundamental way indexed products of cardinals with indexed sums of cardinals :

Theorem 4.39. *If $\kappa_i < \lambda_i$ for each $i \in I$, then $\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i$.*

Proof. The reader can find a proof in [7] (Theorem 5.10, p. 54). ■

Next, we state a result which relates exponentiation with the other cardinal arithmetical operations.

Theorem 4.40. *Let κ, λ, σ be any cardinals. Then, the following holds :*

$$\kappa^{\lambda+\sigma} = \kappa^\lambda.\kappa^\sigma \text{ and } (\kappa^\lambda)^\sigma = \kappa^{\lambda.\sigma}$$

Proof. The reader can find a proof in [6] (Lemma 10.27, p. 32). ■

It is now appropriate to formulate the Continuum Hypothesis :

Continuum Hypothesis (CH) : $2^{\aleph_0} = \aleph_1$

Cantor's original ideas about the theory of ordinals started a revolution in mathematics. The counter intuitive ideas concerning the existence of *different infinities* and also the non-constructivist aspect of his work, lead to many of his contemporary proeminent mathematicians, such as Kronecker and Poincare to consider his ideas heretic. However, in 1900 Hilbert included the CH in his famous list containing what he considered to be the most important problems in mathematics.

God made the integers, all the rest is the work of man. - Kronecker

No one shall expel us from the paradise that Cantor has created. - Hilbert

In 1940, Kurt Godel proved that the negation of the CH cannot be proved in ZFC, while Cohen proved in 1963 that the CH cannot be proved in ZFC, using the method of *forcing* which he developed in order to obtain extension of models. Cohen won a Fields Medal in 1966 for his contributions, establishing the CH as a statement which is independent from ZFC.

In the remaining of this section, we introduce the notion of *cofinality* and as an application, we shall prove in ZFC that $2^{\aleph_0} \neq \aleph_\omega$ (or actually, $2^{\aleph_0} \neq \aleph_\alpha$ for any α with cofinality ω).

Definition 4.41. Let $(P, <)$ be a partially ordered set and $A \subseteq P$. We say that A is **cofinal** in P if for every $b \in P$ there is some $a \in A$ such that $b \leq a$.

Definition 4.42. Let α and β be two ordinals and consider a map $f : \alpha \to \beta$. We say that f maps α cofinally if $f(\alpha)$ is cofinal in β. One then defines the **cofinality** of β - $cf(\beta)$ - to be the least α such that there exists a map from α cofinally into β.

Clearly, $cf(\beta) \leq \beta$ and, if β is a successor ordinal, then $cf(\beta) = 1$. Hence it is only interesting to consider the cofinality of limit ordinals. In this case, it is usual to use the following (equivalent) definition :

Definition 4.43. Let β be a limit ordinal. A sequence $\langle \alpha_\xi : \xi < \beta \rangle$ in α is said to be cofinal if $\lim_{\xi \to \beta} \alpha_\xi = \alpha$. Let α be an infinite limit ordinal and $\mathcal{C}$ be the class of limit ordinals β for which there exists an increasing β-sequence in α, say $\langle \alpha_\xi : \xi < \beta \rangle$, which is cofinal in β. We then define the cofinality of α to be :

$$cf(\alpha) = \min \mathcal{C}$$

Definition 4.44. β is said to be **regular** if β is a limit ordinal such that $cf(\beta) = \beta$. Otherwise, β is said to be **singular**.

Remark 4.45. Note once again that, by definition, $cf(\alpha) \leq \alpha$. On the other hand, if $\langle \alpha_\xi : \xi < \beta \rangle$ is cofinal in α and $\langle \xi(\nu) : \nu < \gamma \rangle$ is cofinal in β, then $\langle \alpha_{\xi(\nu)} : \nu < \gamma \rangle$ is cofinal in α. Hence, it follows that $cf(cf(\alpha)) = cf(\alpha)$, so $cf(\alpha)$ is regular.
Now, let $f : \alpha \to \beta$ be a strictly increasing cofinal map, with α a limit ordinal. It follows that $cf(\alpha) = cf(\beta)$. Thus, if α is a limit ordinal, it follows that $cf(\omega_\alpha) = cf(\alpha)$. In particular, ω_ω is singular.

An useful result to determine whether or not an ordinal (in this case, a cardinal) is singular, is the following :

Theorem 4.46. *Let κ be an infinite cardinal. Then, κ is singular if and only if there is a cardinal $\lambda < \kappa$ and a family $\{S_\xi : \xi < \lambda\}$ of subsets of κ such that $|S_\xi| < \kappa$ and $\kappa = \bigcup_{\xi<\lambda} S_\xi$. The least cardinal λ that satisfies the above condition is $cf(\kappa)$.*

Proof. If κ is singular, then there is an increasing sequence $\{\alpha_\xi : \xi < cf(\kappa)\}$ with $\lim_\xi \alpha_\xi = \kappa$ and we simply take $S_\xi = \alpha_\xi$.
Conversely, let $\lambda < \kappa$ be the least cardinal such that there is some family $\{S_\xi : \xi < \lambda\}$ with $\kappa = \bigcup_{\xi<\lambda} S_\xi$ and $|S_\xi| < \kappa$. Let β_ξ be the order type of $\bigcup_{\nu<\xi} S_\nu$ for each $\xi < \lambda$. The sequence $\langle \beta_\xi : \xi < \lambda \rangle$ is nondecreasing and by minimality of κ, $\beta_\xi < \kappa$ for all $\xi < \kappa$. It is then enough to show that $\lim_{\xi\to\lambda} \beta_\xi = \kappa$: Let $\beta = \lim_{\xi\to\lambda} \beta_\xi$. We define an injective map :

$$f : \kappa = \bigcup_{\xi<\lambda} S_\xi \hookrightarrow \lambda \times \beta$$

such that for all $\alpha \in \kappa$, $f(\alpha) = (\xi, \gamma)$ with ξ the least index i such that $\alpha \in S_i$ and γ the order type of $S_\xi \cap \alpha$.
Since f is injective, it follows that $\lambda|\beta| \geq \kappa$. Thus, $|\beta| \geq \kappa$ since $\lambda < \kappa$. On the other hand, since $\beta_\xi < \kappa$, it follows that $|\beta| \leq \kappa$ and we can conclude that $\beta = \kappa$. ■

We finish this section with an interesting application of the notion of cofinality to the CH :

Theorem 4.47. *Let κ be an infinite cardinal. Then, $\kappa^{cf(\kappa)} > \kappa$.*

Proof. Let $\kappa_i < \kappa$ for $i < cf(\kappa)$ such that $\sum_{i<cf(\kappa)} \kappa_i = \kappa$. Then, by Theorems 4.38 and 4.39 it follows that :

$$\kappa = \sum_{i<cf(\kappa)} \kappa_i < \prod_{i<cf(\kappa)} \kappa = \kappa^{cf(\kappa)}$$

■

Corollary 4.48. $cf(2^{\aleph_\alpha}) > \aleph_\alpha$

In particular, it follows that it is provable in ZFC that $2^{\aleph_0} \neq \aleph_\omega$ or that $2^{\aleph_0} \neq \aleph_{\omega_1+\omega}$.

4.1.3 Models and independence

In this section, we aim to explain what it means for a statement to be independent of a theory (for instance, the CH is independent from ZFC). We will mostly state terminology and try to convey some flavour around these ideas, not entering into any cumbersome technical details. We follow mainly [29].

We will use the language $\mathcal{L}(\in, =)$ of set theory, which we shall define now :

The **alphabet** of $\mathcal{L}(\in, =)$ is a finite collection of symbols, which include :

$$\vee, \neg, (,), \exists, =, \in$$

The remaining symbols of the alphabet are **variable symbols** such as :

$$x_1, x_2, x_3 \text{ or } x, y, z$$

The language $\mathcal{L}(\in, =)$ consists of certain finite sequences $\langle a_1, ..., a_n \rangle$ where each a_i belongs to the alphabet. These finite sequences are the **formulae** of the language. The **syntax** of $\mathcal{L}(\in, =)$ specifies which sequences are formulae, i.e is a list of rules which state how formulae are formed :

(i) (**Atomic rule**) : For all variable symbols x_i and x_j, $x_i = x_j$ and $x_i \in x_j$ are formulae.

(ii) (**Connective rule**) : If φ and Ψ are formulae, then $(\varphi) \vee (\Psi)$ and $\neg(\varphi)$ are formulae.

(iii) (**Quantifier rule**) : If φ is a formula and x_i is a variable symbol, then $\exists x_i(\varphi)$ is a formula.

Definition 4.49. The **language** $\mathcal{L}(\in, =)$ is the smallest set of finite sequences of the alphabet which is closed under application of any of the rules (i)-(iii).

Remark 4.50. We often use the following abbreviations :

(i) $(\varphi) \wedge (\Psi)$, for $\neg((\neg(\varphi)) \vee (\neg(\Psi)))$

(ii) $\forall x_i(\varphi)$, for $\neg(\exists x_i(\neg(\varphi)))$

(iii) $\varphi \Rightarrow \Psi$, for $((\neg(\varphi)) \vee (\Psi))$

(iv) $\varphi \iff \Psi$, for $(\varphi \Rightarrow \Psi) \wedge (\Psi \Rightarrow \varphi)$

(v) $x_i \notin x_j$, for $\neg(x_i \in x_j)$

Given a formula φ of $\mathcal{L}(\in, =)$, the **subformulae** of φ are precisely those formulae which arise in the construction of φ. For instance, let φ be :

$$(\exists x_5(x_2 \in x_7)) \vee (\exists x_7(x_1 = x_7))$$

Then, an example of a subformulae would be $\exists x_5(x_2 \in x_7)$.

Now let x_i be a variable. The **scope** of a particular occurrence of $\exists x_i$ in a formula φ is the unique subformula of φ which begins with that occurrence of $\exists x_i$. For instance, the scope of $\exists x_5$ is $\exists x_5(x_2 \in x_7)$.
An occurrence of a variable x_i in a formula φ is **free** if it does not belong to the scope of occurrence of $\exists x_i$ in φ. Otherwise, we say that the occurrence of the variable x_i is **bound**. For instance, the first occurrence of x_7 is free and the second and third occurrences of x_7 are bound.

Definition 4.51. A variable is called **free** if there is at least on free occurrence of it and is **bound** if every occurrence of it is bound. A **sentence** is a formula which has no free variables and a **theory** in $\mathcal{L}(\in, =)$ is a set of sentences of $\mathcal{L}(\in, =)$.

How does a theory prove a formula ? The language $\mathcal{L}(\in, =)$ contains certain formulae which are defined as **logical axioms**. The logical axioms are considered to be *trivially true* and there is some freedom in their choice. For instance, examples of logical axioms which are commonly used would be :

$$\varphi \Rightarrow \varphi,\ \varphi \Rightarrow (\Psi \Rightarrow (\varphi \wedge \Psi)),\ \forall x_i(\varphi) \Rightarrow \varphi \text{ or } \varphi \vee \neg(\varphi)$$

The set of logical axioms also include formulae which capture our intuition that $=$ should be an equivalence relation which $\in$ respects. For instance, formulae like : $(x = y) \wedge (y \in z) \Rightarrow (x \in z)$.

Remark 4.52. Systems of intuitionistic logic do not include the law of third excluded $\varphi \vee \neg(\varphi)$. Even in *common language*, the Liar's Paradox is an example of a statement that is neither true or false.

A **rule of inference** is a rule to derive a formula from a collection of formulae of the language. We have the following :

$$(\textbf{Modus Ponens}) : \text{From } \{\varphi, \varphi \Rightarrow \Psi\}, \text{ derive } \Psi.$$

Definition 4.53. Let T be a theory in $\mathcal{L}(\in, =)$ and φ a formula. Then, T **proves** φ if there is a finite sequence $\langle \varphi_1, ..., \varphi_n \rangle$ of formulae of $\mathcal{L}(\in, =)$ such that $\varphi_n = \varphi$ and either φ_i is an element of T, a logical axiom or obtained by Modus Ponens from two formulae in $\{\varphi_1, ..., \varphi_{i-1}\}$. We write $T \vdash \varphi$ if T proves φ or $T \nvdash \varphi$ if T does not prove φ.

Definition 4.54. Let T be a theory in $\mathcal{L}(\in, =)$. Then, T is **inconsistent** if there is a sentence φ such that $T \vdash \varphi \wedge \neg(\varphi)$. Otherwise, T is said to be **consistent**.

Remark 4.55. If T is inconsistent, then $T \vdash \Psi$ for every sentence Ψ. This is sometimes called the Principle of Explosion. Indeed,

(1) Assume $\varphi \wedge \neg(\varphi)$

(2) From (1) and by conjunction, derive φ

(3) From (1) and by conjunction, derive $\neg(\varphi)$

(4) From (2) and by disjunction, derive $\varphi \vee \Psi$

(5) From (3) and and (4), by disjunction syllogism, derive Ψ

(6) From (5), derive $(\varphi \wedge \neg(\varphi)) \Rightarrow \Psi$

(7) By Modus Ponens applied to (1) and (6), derive Ψ

Hence, $\forall\varphi\forall\Psi : (\varphi \wedge \neg(\varphi)) \vdash \Psi$ and we work with consistent theories only.

We had explained what a proof means in the syntax universe. Now, we shift our focus and try to capture the notion of *truth* in the semantic domain. In order to do so, we introduce models. It follows from Godel's Completeness Theorem that these concepts are related in an intimate manner.

Definition 4.56. A **model** $\mathcal{M}$ is a pair (M, E), where M is a non-empty set and E is a subset of $M \times M$. The model is said to be countable when M is a countable set.

Let $\mathcal{M} = (M, E_M)$ and $\mathcal{N} = (N, E_N)$ be two models. Then, $\mathcal{M}$ is a **submodel** of $\mathcal{N}$ (or, $\mathcal{N}$ is an **extension** of $\mathcal{M}$) if $M \subseteq N$ and $E_N \cap M \times M = E_M$. For instance, $(\mathbb{Q}, <)$ is an extension of $(\mathbb{Z}, <)$.

We shall define now, by induction on the length of formulae φ, the **truth** of a formula φ in a model $\mathcal{M}$. For elements $a_i \in M$, we write $\mathcal{M} \models \varphi[a_1, ..., a_n]$ if $\mathcal{M}$ **satisfies** $\varphi[a_1, ..., a_n]$, i.e if φ is true at $(a_1, ..., a_n)$ in $\mathcal{M}$. Otherwise, we write $\mathcal{M} \not\models \varphi[a_1, ..., a_n]$. We set the following rules :

Definition 4.57. (i) Let x_i and x_j be variables. Then :

$$\mathcal{M} \models (x_i = x_j)[a_1, ..., a_n], \text{ if } a_i = a_j$$

$$\mathcal{M} \models (x_i \in x_j)[a_1, ..., a_n], \text{ if } (a_i, a_j) \in E$$

(ii) Let φ and Ψ be formulae. Then :

$$\mathcal{M} \models (\varphi \vee \Psi)[a_1, ..., a_n], \text{ if } \mathcal{M} \models \varphi[a_1, ..., a_n] \text{ or } \mathcal{M} \models \Psi[a_1, ..., a_n]$$

(iii) $\mathcal{M} \models (\neg\varphi)[a_1, ..., a_n]$, if $\mathcal{M} \not\models \varphi[a_1, ..., a_n]$

(iv) Let φ be a formula and x_i a variable. Then :

$$\mathcal{M} \models (\exists x_i \varphi)[a_1, ..., a_n], \text{ if for some } b \in M,\ \mathcal{M} \models \varphi[a_1, ..., a_{i-1}, b, a_{i+1}, ..., a_n]$$

Definition 4.58. Let $\mathcal{M}$ be a model. The **theory of** $\mathcal{M}$ is the set of sentences φ in $\mathcal{L}(\in, =)$ such that $\mathcal{M} \models \varphi$. We denote it by $Th(\mathcal{M})$. We say that $\mathcal{M}$ is a **model of a theory** T if $T \subseteq Th(\mathcal{M})$ and we write $\mathcal{M} \models T$.

We are mostly interested in the ZFC theory, so T consists of the set of sentences which are usually called ZFC axioms (plus, the logical axioms). As an example, let's consider two of these axioms :

$$\text{Axiom of Pairing} : \forall x \forall y \exists z((x \in z) \wedge (y \in z))$$

$$\text{Axiom of Foundation} : \forall x \exists y \forall z((z \notin x) \vee (y \in x \wedge (z \in y \Rightarrow z \notin x)))$$

Consider $\mathcal{M} = (\mathbb{R}, <)$. On one hand, $\mathcal{M} \models$ Pairing, since given $x, y \in \mathbb{R}$, there is some $z \in \mathbb{R}$ such that $x < z$ and $y < z$. On the other hand, $\mathcal{M} \not\models$ Foundation since for a given $x \in \mathbb{R}$ it is not true that there is some $y < x$ such that for each $z \in \mathbb{R}$, if $z < y$ then $z \not< x$.
However, the submodel $\mathcal{N} = (\omega, <)$ is such that $\mathcal{N} \models$ Foundation : let $x \in \omega$. If $x = 0$, then $z \not< x$ and if $x \geq 1$, we can choose $y = 0$.

In order to get some intuition on models, it may be useful to make the following analogy : A group is a set G with a binary operation $*$ satisfying axioms like :

$$\exists e \in G \forall x \in G : x * e = e * x = x$$

A model for ZFC can be seen as a set M together with a relation E which satisfies each one of the ZFC axioms. In this case, E is usually intended to represent $\in$.

One can construct non-abelian groups which are simply groups where the following formula does not hold :

$$\forall x \forall y : x * y = y * x$$

Similarly, we may try to construct a model for ZFC which is such that a certain formula (for instance, the statement of the CH) does not hold. On the other hand, one can provide an example of an abelian group and similarly, one may try to construct a model for ZFC where a certain formula holds. In a sense, it is fair to say that *theory of models is essentially the theory of mathematics multiverses.*

Definition 4.59. Let T be a theory in $\mathcal{L}(\in, =)$ We define :

$$Thm(T) = \{\varphi \in \mathcal{L}(\in, =) : \varphi \text{ is a sentence and } T \vdash \varphi\}$$

A sentence φ is said to be **independent** of a theory T if $\varphi \notin Thm(T)$ and $(\neg\varphi) \notin Thm(T)$.

Kurt Godel, arguably one of the most influential mathematicians of the 20^{th} century is widely known for his Incompleteness Theorems, which had an immeasurable impact (both mathematical and metamathematical). However, his Completeness Theorem is also of cornerstone importance, identifying the notion of *proof* and *truth*.

Theorem 4.60. ***(Godel's Completeness Theorem)*** *Let T be a theory of $\mathcal{L}(\in,=)$. The following are equivalent :*

1. *T is consistent*
2. *T has a model*
3. *T has a countable model*

Remark 4.61. If the truth of a formula in each submodel $\mathcal{N}$ of $\mathcal{M}$ follows from its truth in $\mathcal{M}$, the formula is said to be **downward absolute**. On the other hand, if the truth of a formula in $\mathcal{N}$ implies the truth of the formula in each extension $\mathcal{M}$ of $\mathcal{N}$, the formula is said to be **upward absolute**. Formulae which have the same truth value in each model are called **absolute**.
ZFC proves the existence of an uncountable set. Assuming that ZFC is consistent, it has a countable model and thus, that set in this model is necessarily countable (even though the model sees it as uncountable). This is called the **Skolem's paradox** and provides an example of a notion which is not absolute - the notion of cardinality. To make things more clear, recall the Power Set Axiom from ZFC :

$$\forall x \exists y \forall z (z \in y \Leftrightarrow (\forall w \in z \Rightarrow w \in x))$$

Let $\mathcal{M}$ be a model of ZFC and $\aleph_0 \in M$. The Power Set Axiom states that y contains every subset of $\mathcal{N}_0$ **that is in** M. If we mimic the proof of Cantor's Theorem in $\mathcal{M}$, then it will follow that y will be *uncountable* in the eyes of $\mathcal{M}$ but if M is countable, there will be bijections between y and $2^{\aleph_0}$ which will be *missing* in $\mathcal{M}$.
Even the notion of subset is not absolute. Indeed, consider $M = \{0, a\}$ with $a = \{\{0\}\}$. Then, $\mathcal{M} \models a \subseteq 0$, but $a \nsubseteq 0$.
In order to have a better control over absolute sentences, it is usual to consider only **transitive** models $\mathcal{M}$ which are such that if $x \in y$ and $y \in M$, then $x \in M$. Among these models, plenty of statements are absolute (such as the notion of *subset*, *pairings*, *relation*, *function*, etc...)

It can be shown that the sentences in the set of the logical axioms of $\mathcal{L}(\in,=)$ are true in every model and moreover, if $\mathcal{M} \models \varphi$ and $\mathcal{M} \models (\varphi \Rightarrow \Psi)$, then $\mathcal{M} \models \Psi$. It follows that if $\mathcal{M} \models T$ and $T \vdash \varphi$, then $\mathcal{M} \models \varphi$.
A way to prove the independence of a certain sentence φ from a theory T (which we will always assume to be consistent) is then to provide two models of T such that φ is true on one of them and $\neg(\varphi)$ is true on the other (like the previous example with abelian and non-abelian groups).

Indeed, assume that ZFC is consistent and that there are two models :

$$\mathcal{M}_1 \models (ZFC + CH)$$

$$\mathcal{M}_2 \models (ZFC + \neg(CH))$$

This establishes the independence of the CH from ZFC. Indeed, suppose that $CH \in Thm(ZFC)$. But then, $\mathcal{M}_2 \models ZFC + CH + \neg(CH)$ contradictiong Theorem 4.60. Similarly, if we assume that $\neg(CH) \in Thm(ZFC)$ we will derive a contradiction.

In 1940, Godel proved that $\neg(CH) \notin Thm(ZFC)$ and in 1963, Cohen proved that $CH \notin Thm(ZFC)$. We end this section with a short list of statements among different fields in mathematics which have been shown to be independent from ZFC :

• **Generalized Continuum Hypothesis** : $2^{\aleph_\alpha} = \aleph_{\alpha+1}$

• **Whitehead Problem** : If G is an abelian group such that $Ext^1(G, \mathbb{Z}) = 0$ does it follows that G is free ?

• **MA(κ), for $\omega < \kappa < 2^\omega$** : Let $\mathbb{P}$ be a c.c.c. partially ordered set and let $\mathcal{D}$ be a family of order dense subsets in $\mathbb{P}$ such that $|\mathcal{D}| \leq \kappa$. Then, there is a filter $\mathcal{F}$ on $\mathbb{P}$ such that $d \cap \mathcal{F} \neq \emptyset$ for each $d \in \mathcal{D}$.

• **Matiyasevich polynomial** : There is a polynomial $p \in \mathbb{Z}[x_1, ..., x_9]$ such that the statement that there are $n_1, ..., n_9 \in \mathbb{Z}$ with $p(n_1, ..., n_9) = 0$ is independent from ZFC.

• **Kaplansky's Conjecture** : Every algebra homomorphism from $C(X)$, with X a compact and Hausdorff space, into any Banach algebra must be continuous.

4.2 Polish spaces

In this section we state several well known results about Polish spaces which are heavily referenced in all previous sections. We often ommit details, as each result contained in this section is quite standard.

Definition 4.62. A topological space X is said to be Polish if it is completely metrizable and second countable.

The definition of a Polish space is probably the simplest, yet general enough class of spaces that include plenty of interesting examples.

Let X be a complete metrizable space, without isolated points. For any $s \in 2^{<\omega}$ we will associate a closed subset of X, starting with $F_\emptyset = X$. Since X has no isolated points, it certainly contains two distinct points and, using regularity of

X, one can find two open sets, $x_0 \in \mathcal{U}_0$ and $x_1 \in \mathcal{U}_1$ such that $\overline{\mathcal{U}_0} \cap \overline{\mathcal{U}_1} = \emptyset$. We take $F_i = \overline{\mathcal{U}_i}$, $i \in 2$. By induction, and essentially repeating the same argument on each $\mathcal{U}_s$, one define a family of closed subsets of X, $\{F_s\}_{s \in 2^{<\omega}}$ such that :

(i) For all $i \in 2$ and $s \in 2^{<\omega}$, $\emptyset \neq F_{s \frown i} \subseteq F_s$

(ii) For each $\alpha \in 2^\omega$, then $\operatorname{diam}(F_{\alpha|_n}) \to 0$ with $n \to \infty$

(iii) If $s \perp t$, then $F_s \cap F_t = \emptyset$

One can then define $C = \{\alpha \in 2^\omega : \forall n (F_{\alpha|_n} \neq \emptyset)\}$, easily seen to be a closed subset. Moreover, since X is complete, the following map is well-defined with $f(\alpha) \neq \emptyset$:

$$f : C \to X, \text{ such that } f(\alpha) = \bigcap_n F_{\alpha|_n}$$

Furthermore, it is easy to check that f is injective and continuous. Therefore, we have proved that :

Theorem 4.63. *Let X be a complete metrizable space without isolated points. Then, X contains a copy of the Cantor set 2^ω.*

Now assume that X is metrizable and with a countable basis, say $\{\mathcal{U}_n\}$. If one takes $Y = \bigcup_n \{\mathcal{U}_n : \mathcal{U}_n \text{ is countable}\}$ and $Z = X \setminus Y$, it follows that :

Theorem 4.64. *Let X be a second countable metrizable space. Then, $X = Y \sqcup Z$ with Y countable and Z closed and without isolated points.*

It follows, by Theorems 4.63 and 4.64 that every uncountable Polish space contains a homeomorph copy of 2^ω. Moreover, if one considers $E \subseteq 2^\omega$ to be the collection of sequences in 2^ω which have infinitely many 0's and 1's, it is not hard to check that E and ω^ω are homeomorphic. Hence :

Theorem 4.65. *Let X be an uncountable Polish space. Then :*

(i) X contains a homeomorph of the Cantor set 2^ω.

(ii) X contains a homeomorph of the Baire space ω^ω.

The next result is often useful and provides a quite pragmatic characterization of subspaces of Polish spaces :

Theorem 4.66. *Let X be a Polish space and $Y \subseteq X$. Then, Y is Polish if and only if Y is a G_δ-set.*

Proof. The reader can check a proof in [1] (Theorem 3.11, p.17). ■

Remark 4.67. It is worth to note that Theorems 4.65 and 4.66, immediately imply that the Continuum Hypothesis holds in ZFC for G_δ subsets of a Polish space. In particular it holds for open and closed sets.

An important class of examples (for instance in section 2.3) is the family of zero dimensional Polish spaces. Suppose that X is a zero dimensional second countable metrizable space, with a basis of clopen sets $\{\mathcal{U}_n\}$. One can define the following :

$$f : X \to 2^\omega \text{ such that } f(x) = (\chi_n(x))_{n\in\omega}$$

where each χ_n is just the characteristic map of $\mathcal{U}_n$. It is not hard to check that f is an embedding. In fact, one has that :

Theorem 4.68. *Let X be a zero dimensional Polish space. Then :*

(i) X is homeomorphic to a closed subspace of ω^ω.

(ii) X is homeomorphic to a G_δ subspace of 2^ω.

Proof. The reader can find a proof in [1] (Theorem 7.8, p.38). ∎

It follows immediately that :

Theorem 4.69. *ω^ω is homeomorphic to a G_δ subset of 2^ω.*

Another useful result which further emphatizes the central role of the space ω^ω is the following :

Theorem 4.70. *Let X be a Polish space. Then, there is a closed subset $F \subseteq \omega^\omega$ and a continuous bijection $f : F \to X$. In particular, if X is non-empty, there is a continuous surjection $g : \omega^\omega \to X$ which extends f.*

Proof. The reader can find a proof in [1] (Theorem 7.9, p.38). ∎

The following result is useful when studying closed subsets of products of discrete spaces :

Theorem 4.71. *Let A be a discrete space and consider closed subsets $F \subseteq G$ of the product space A^ω. Then, there is a continuous map $f : G \to F$ such that is the identity map when restricted to F.*

Proof. The reader can check a proof in [1] (Proposition 2.8, p.9). ∎

A continuous map $f : X \to Y$ between two topological spaces X and Y is such that the preimage of an open set in Y is an open set in X. Similarly, if (X, Σ_1) and (Y, Σ_2) are measure spaces, a measurable map $f : X \to Y$ is such that the preimage of an element in Σ_2 is still an element in Σ_1. We say that f is a bimeasurable map if f is measurable and also takes elements in Σ_1 in elements in Σ_2. In particular, if $f : (X, \mathcal{B}(X)) \to (Y, \mathcal{B}(Y))$ is a bijection and bimeasurable, we say that f is a Borel isomorphism.

Definition 4.72. A measure space (X, Σ) is said to be a Borel standard space if it is Borel isomorphic to some $(Y, \mathcal{B}(Y))$, with Y a Polish space. Equivalently, if there is some Polish topology $\mathcal{T}$ on X such that $\mathcal{B}(X) = \Sigma$.

Consider a Polish space $(X, \mathcal{T})$ with $A \subseteq X$ a Borel set. Then, there is a Polish topology $\mathcal{T}_A \supseteq \mathcal{T}$ on X such that A is clopen, with $\mathcal{B}(\mathcal{T}) = \mathcal{B}(\mathcal{T}_A)$.
Indeed, we first observe that if $F \subseteq X$ is closed, then the topology generated by $\mathcal{T} \cup \{F\}$ - call it $\mathcal{F}_T$ - is such that $\mathcal{T}_F$ is Polish and with F clopen and $\mathcal{B}(\mathcal{T}) = \mathcal{B}(\mathcal{T}_F)$.
Moreover, one can check that if $\{\mathcal{T}_n\}$ is a family of Polish topologies on X with $\mathcal{T} \subseteq \mathcal{T}_n$, then the topology $\mathcal{T}_\infty$ generated by $\bigcup_n \mathcal{T}_n$ is also Polish and furthermore, if $\mathcal{T}_n \subseteq \mathcal{B}(\mathcal{T})$, then $\mathcal{B}(\mathcal{T}) = \mathcal{B}(\mathcal{T}_\infty)$.
Finally, let $\mathcal{C}$ be the collection of subsets of $A \subseteq X$ for which there exists a Polish topology $\mathcal{T}_A$ on which A is clopen and $\mathcal{B}(\mathcal{T}) = \mathcal{B}(\mathcal{T}_A)$. It follows by the previous comments that $\mathcal{C}$ is a σ-algebra which contains $\mathcal{T}$ and we are done.
Hence, it follows by Remark 4.67 that :

Theorem 4.73. *Let X be a Polish space. Then, the Continuum Hypothesis holds for Borel sets of X.*

Another (very) important result on standard Borel spaces is a remarkably useful result that allows reducing the study of a standard Borel space to a *simpler* one. It is usually used to reduce the study of arbitrary standard Borel spaces either to 2^ω or ω^ω.

Theorem 4.74. *Any two uncountable standard Borel spaces are Borel isomorphic.*

Proof. The reader can check a proof in [2] (Theorem 3.3.13, p.99). ■

We finish this section with a statement about the cardinality of continuous maps from ω^ω to itself and some remarks on the Cantor-Bendixson derivative. The result about continuity, of its own interest, is also used in section 2.3 in order to describe hierarchies of zero dimensional Polish spaces

Theorem 4.75. *There are exactly $2^{\aleph_0}$ continuous maps from ω^ω to itself.*

Proof. Let $A \subseteq \{f : \omega^{<\omega} \to \omega^{<\omega}\}$ be such that $\varphi \in A$ if and only if whenever $s \subseteq t$ then $\varphi(s) \subseteq \varphi(t)$ and, if $x \in \omega^\omega$, then $\lim_n |\varphi(x|_n)| = \infty$ with $|.|$ denoting the length of an element in $\omega^{<\omega}$. Moreover, let B be the set of all continuous maps $f : \omega^\omega \to \omega^\omega$. We prove that $|A| = |B|$.
Assume for a moment that both sets A and B have the same cardinality. Since each constant map from ω^ω to itself is continuous, $|B| \geq 2^{\aleph_0}$. On the other hand, note that $\omega^{<\omega}$ is countable and thus, $|A| \leq 2^{\aleph_0}$. Hence, it follows that $|B| = 2^{\aleph_0}$, as we want.
In order to prove that $|A| = |B|$, by Cantor-Schroeder Theorem it is enough to construct injective maps $\Lambda : A \to B$ and $\Psi : B \to A$. Let :

$$\Lambda(\varphi)(x) = \bigcup_n \varphi(x|_n)$$

Since $\varphi \in A$, it is easily seen that Λ is well-defined and injective.

On the other hand, let $f \in B$, $x \in \omega^\omega$ and $m \in \omega$. Then, there is certainly some $m' \in \omega$ such that $x \in \Sigma(x|_{m'}) \subseteq f^{-1}(\Sigma(f(x)|_m))$, due to continuity of f. For an element $s \in \omega^{<\omega}$ we let $m_s = \sup\{m \in \omega : m \leq |s|\}$ and define :

$$\Psi(f)(s) = f(s \frown 0^\omega)|_{m_s}$$

It is easy to check that Ψ is a well defined injective map. ■

In Section 3.3 the concept of Cantor-Bendixson derivative is used. One could define it in a more general setting however, and since it appears in this thesis only in that context, we shall ommit any such generalizations.

Consider $E \subseteq \mathbb{R}$, any closed set and let $E' = \{x \in E : \ x \text{ is a limit point of } E\}$. We define by transfinite recursion the following sets :

$$E^0 = E,\ E^{\alpha+1} = (E^\alpha)' \text{ for every ordinal } \alpha \text{ and,}$$

$$E^\beta = \bigcap_{\alpha<\beta} E^\alpha, \text{ if } \beta \text{ is a limit ordinal}$$

Thus, we produce a decreasing chain of closed sets :

$$E \supseteq E^1 \supseteq ... \supseteq E^\alpha \supseteq ...$$

Since $\mathbb{R}$ is second countable, let's fix some countable basis $\{\mathcal{V}_n\}$ and suppose that $\{F_\alpha\}_{\alpha\geq 0}$ is any decreasing chain of closed subsets indexed by ordinals α. Let $A_\alpha = \{n : \mathcal{V}_n \cap F_\alpha = \emptyset\}$ so that if $\alpha \leq \beta$, then $A_\alpha \subseteq A_\beta$. If $A_\alpha \subsetneq A_{\alpha+1}$ for all countable ordinals, let $f(\alpha)$ be the least n such that $n \in A_{\alpha+1} \setminus A_\alpha$. Then, $f : \omega_1 \to \omega$ is an injective map which is a contradiction. Thus, we conclude that there is some countable ordinal α_0 such that for all $\alpha \geq \alpha_0$, $A_{\alpha_0} = A_\alpha$ and hence, $F_\alpha = F_{\alpha_0}$. If we apply this to the chain of Cantor-Bendixson derivatives:

Theorem 4.76. *Let $E \subseteq \mathbb{R}$ be a closed set. Then, there is the smallest countable α such that $E^\alpha = E^\beta$ for all $\beta \geq \alpha$. This is called the Cantor-Bendixson rank of E*

Remark 4.77. If $\alpha \in \omega_1$ is the Cantor-Bendixson rank of $E \subseteq \mathbb{R}$, it is usual to denote $E^\infty = E^\alpha$.

We end this section with the following small, but useful observation :

Theorem 4.78. *Let $E \subseteq \mathbb{R}$ be a closed subset. Then, $E \setminus E^\infty$ is countable. In particular, E is countable if and only if $E^\infty = \emptyset$.*

Proof. Let $rk(E)$ be the Cantor-Bendixson rank of E and suppose that $x \in E \setminus E^\infty$. It follows that there is some countable $\alpha < rk(E)$ such that $x \in E^\alpha \setminus E^{\alpha+1}$. Thus, it is enough to note that given any closed set F, $F \setminus F'$ is countable. Indeed, let $\{\mathcal{U}_n\}$ be a countable basis. If $x \in F \setminus F'$, then there is some n such that $F \cap \mathcal{U}_n = \{x\}$ and we note that $F \setminus F' = \bigcup\{F \cap \mathcal{U}_n : F \cap \mathcal{U}_n \text{ is a singleton}\}$. ■

4.3 Trees

In this subsection we recall some elementary definitions about trees. These are combinatorial objects with great use in descriptive set theory methods.

Let A be any set. Given $s, t \in A^{<\omega}$, we say that s is an initial segment of t (written as $s \subseteq t$) if there is some $n \in \omega$ such that $s = t|_n$.
A **tree** T on A is a subset $T \subseteq A^{<\omega}$ which is closed under initial segments, i.e if $t \in T$ and $s \subseteq t$, then $s \in T$. An element $t \in T$ is often called a node.

Given a tree $T \subseteq A^{<\omega}$, we define its body $[T]$ to be :

$$[T] = \{x \in A^{\omega} : \forall n(x|_n \in T)\}$$

If $[T] = \emptyset$, the tree is said to be **well-founded**. Otherwise, T is said to be **ill-founded**. Relying on reduction theorems for Polish spaces, it is often sufficient to consider trees on ω, although this may not be the case on a more game-theoretic flavoured context (as in Section 2).

A tree is said to be **pruned** if every node $s \in T$ has a proper extension $t \supsetneq s$, with $t \in T$. A node is called terminal, if it admits no extension.

Given a tree $T \subseteq A^{<\omega}$ and a node $s \in T$ it is sometimes convenient to consider the following tree :

$$T_s = \{t \in A^{<\omega} : s \frown t \in T\}$$

In this context, $t \frown s$ denotes the concatenation of $t = (t_i)_{i=0}^{n}$ and $s = (s_j)_{j=0}^{m}$, i.e $t \frown s = (t_0, ..., t_n, s_0, ..., s_m)$.

Now, suppose that T is a well founded tree. One can define recursively a map $\rho_T : T \to \mathbf{On}$ as follows :

$$\text{For any } s \in T,\ \rho_T(s) = \sup\{\rho_T(t) + 1, t \supsetneq s, t \in T\}$$

It should be clear that $\rho_T(s) = \sup\{\rho_T(s \frown a) + 1 : s \frown a \in T\}$. This strenghtens the intuition that the map ρ_T measures how *deep* a node is on a tree.

We define the **rank** of T as $\rho(T) := \rho_T(\emptyset)$. For instance, if we consider the tree $T = \{\emptyset\} \bigcup \{i \frown 0^j, j \leq i, i \in \omega\}$, then $\rho(T) = \omega$.

In section 1.3, we will actually use the notion of rank for ill-founded trees on ω. If $T \subseteq \omega^{<\omega}$ is a ill-founded tree, then we slightly modify the definition of the map ρ_T : For $s \in \omega^{<\omega}$, then

$$\rho_T(s) = \begin{cases} 0 & \text{, if } s \notin T \\ \rho_{T_s}(\emptyset) & \text{, if } s \in T \text{ and } T_s \text{ is well founded} \\ \omega_1 & \text{, otherwise} \end{cases}$$

We note that T is well-founded if and only if $\rho_T(\emptyset) < \omega_1$.

Using trees, one can obtain a useful characterization of (co)analytic sets of ω^ω. This is the content of Proposition 1.36. The bijection between closed subsets of a product of a discrete space and pruned trees plays an important role. Indeed, let A be a non empty set with the discrete topology and consider its product space A^ω. Furthermore, let $C \subseteq A^\omega$ be a closed set. It is easy to see that :

$$T = \{\alpha|_k : \alpha \in C, k \in \omega\}$$

is a pruned tree such that $[T] = C$. Conversely, it is immediate that given a tree T on A, $A^\omega \setminus [T]$ is an open set. Hence, we have the following :

Theorem 4.79. *Let A be any non-empty set endowed with the discrete topology and consider the product space A^ω. Let $C \subset A^\omega$. Then, C is closed if and only if $C = [T]$ for some tree T on A.*

4.4 $\mathcal{K}(X)$ and $\mathcal{F}(X)$

Let X be a topological space and let $\mathcal{K}(X)$ be the set of all compact subsets of X. The Vietoris topology on $\mathcal{K}(X)$ is the topology generated by the sets of the form :

$$\{K \in \mathcal{K}(X) : K \subseteq \mathcal{U}\} \text{ and}$$

$$\{K \in \mathcal{K}(X) : K \cap \mathcal{U} \neq \emptyset\}, \text{ for an open set } \mathcal{U} \subseteq X$$

A basis for this topology is given by sets of the form :

$$\{K \in \mathcal{K}(X) : K \subseteq \mathcal{U}_0, K \cap \mathcal{U}_1 \neq \emptyset, ..., K \cap \mathcal{U}_n \neq \emptyset\}, \text{ for } \mathcal{U}_i \subseteq X \text{ open sets.}$$

Theorem 4.80. *Let X be a topological space. Then :*

(i) If X is metrizable, so is $\mathcal{K}(X)$.

(ii) If X is separable, so is $\mathcal{K}(X)$.

(iii) If X is completely metrizable, so is $\mathcal{K}(X)$.

(iv) If X is compact and metrizable, so is $\mathcal{K}(X)$.

Corollary 4.81. If X is a Polish space, then so is $\mathcal{K}(X)$.

Theorem 4.82. *Let X be a metrizable space. Then, the following hold :*

(i) *The set* $\{(x,K) : x \in K\}$ *is closed in* $X \times \mathcal{K}(X)$.

(ii) *The sets* $\{(K,L) : K \subseteq L\}$ *and* $\{(K,L) : K \cap L \neq \emptyset\}$ *are closed in* $\mathcal{K}(X) \times \mathcal{K}(X)$.

(iii) *The map* $F : \mathcal{K}(X)^2 \to \mathcal{K}(X)$ *such that* $F(K,L) = K \cup L$ *is continuous.*

(iv) *The map* $G : \mathcal{K}(\mathcal{K}(X)) \to \mathcal{K}(X)$ *such that* $G(C) = \bigcup\{K : K \in C\}$ *is continuous.*

(v) *If* Y *is metrizable and* $f : X \to Y$ *is continuous, then the map* $g : \mathcal{K}(X) \to \mathcal{K}(Y)$ *such that* $g(C) = f(C)$ *is continuous.*

(vi) *The set* $K_{fin} = \{K \in \mathcal{K}(X) : K$ *is finite*$\}$ *is* F_σ *in* $\mathcal{K}(X)$*, the set* $K_{per} = \{K \in \mathcal{K}(X) : K$ *is perfect*$\}$ *is* G_δ *in* $\mathcal{K}(X)$ *and if* $A \subseteq X$ *is closed/open/*G_δ*, then so is the set* $K_A = \{K \in \mathcal{K}(X) :\subset A\}$ *(respectively) in* $\mathcal{K}(X)$.

Let X be a topological space and let $\mathcal{F}(X)$ be the set of all closed subsets of X. One can consider the σ-algebra on $\mathcal{F}(X)$ generated by the following sets :

$$\{F \in \mathcal{F}(X) : F \cap \mathcal{U} \neq \emptyset\}, \text{ for } \mathcal{U} \subseteq X \text{ open set}$$

The set $\mathcal{F}(X)$ endowed with this σ-algebra is called the Effros-Borel space.

Theorem 4.83. *Let* X *be a Polish space. Then :*

(i) *The Effros Borel space* $\mathcal{F}(X)$ *is a standard Borel space.*

(ii) *If* X *is compact, then the Effros Borel space is induced by the Vietoris topology on* $\mathcal{K}(X) = \mathcal{F}(X)$.

Theorem 4.84. *Let* X *be a Polish space. Then :*

(i) $\mathcal{K}(X)$ *is a Borel set in* $\mathcal{F}(X)$.

(ii) *The set* $\{(F_1,F_2) : F_1 \subseteq F_2\}$ *is Borel in* $\mathcal{F}(X)^2$.

(iii) *The map* $F : \mathcal{F}(X)^2 \to \mathcal{F}(X)$ *such that* $F(F_1,F_2) = F_1 \cup F_2$ *is Borel.*

(iv) *If* Y *is a Polish space and* $f : X \to Y$ *is continuous, then the maps* $g : \mathcal{F}(X) \to \mathcal{F}(Y)$ *and* $h : \mathcal{F}(X) \times \mathcal{F}(Y) \to \mathcal{F}(X \times Y)$ *such that* $g(F) = \overline{f(F)}$ *and* $h((F_1,F_2)) = F_1 \times F_2$*, are continuous.*

We end this section by introducing another topology of interest on $\mathcal{F}(X)$, the Wijsman topology. Let X be a metrizable space, with a compatible metric d and consider, for each $x \in X$ the following map $\varphi_x : \mathcal{F}(X) \to \mathbb{R}$ given by $A \mapsto d(x,A)$. The weak topology generated by the family $\{\varphi_x\}_{x \in X}$ on $\mathcal{F}(X)$, is called the Wijsman topology and will be denoted by $\mathcal{W}$. The following result establishes an important relation between the Wijsman topology and the Effros Borel space :

Theorem 4.85. *Let* X *be a Polish space. Then, the space* $(\mathcal{F}(X) \setminus \{\emptyset\}, \mathcal{W})$ *is Polish and furthermore, the Effros Borel space on* $\mathcal{F}(X) \setminus \{\emptyset\}$ *coincides with the Borel space induced by the Wijsman topology.*

4.5 An application of Martin's Axiom : generalization of Bari Theorem

In this section we introduce the Martin's Axiom $MA(\kappa)$ and working within $ZFC + \neg CH + MA(\kappa)$, we generalize Theorems 3.17 and 3.33. More concretely, we prove the following ($\omega \leq \kappa < 2^\omega$) :

Theorem 4.86. *Assume $MA(\kappa)$. Let $\{E_\alpha\}_{\alpha\in\kappa}$ be a family of closed sets of uniqueness. Then, $E = \bigcup_{\alpha\in\kappa} E_\alpha$ is a set of uniqueness.*

Corollary 4.87. Assume $MA(\kappa)$. Let E be any set such that $|E| = \kappa$, for $\kappa < 2^\omega$. Then, E is a set of uniqueness.

Remark 4.88. Theorem 4.86 is a generalization of Bari Theorem (Theorem 3.33), which asserts that a countable union of closed sets of uniqueness is a set of uniqueness. On the other hand, Corollary 4.87 is a generalization of Cantor Theorem, which asserts that any countable set is a set of uniqueness. [13]

4.5.1 Martin's Axiom

For any cardinal $\omega \leq \kappa < 2^\omega$, $MA(\kappa)$ is the following statement :

Let $(\mathbb{P}, \leq)$ be a non-empty c.c.c. partial order and $\mathcal{D}$ a family of $\leq \kappa$ (order) dense subsets of $\mathbb{P}$. Then, there is a filter G in $\mathbb{P}$ such that $G \cap d \neq \emptyset$ for any $d \in \mathcal{D}$.

Theorem 4.89. *$MA(\omega)$ holds in ZFC.*

Proof. Let $\{D_n\}$ be a family of dense subsets of $(\mathbb{P}, \leq)$ (not necessary to have the c.c.c). Choose any $d_1 \in D_1$. Since D_2 is dense, there is some $d_2 \in D_2$ such that $d_2 \leq d_1$. We proceed by induction and define a set $S = \{d_n\}$ with each $d_n \in D_n$ and such that $d_{n+1} \leq d_n$ for all n. We then consider the filter G generated by S. ■

On the other hand, if we allow $\kappa = 2^\omega$ in our definition of $MA(\kappa)$, we get a statement which is inconsistent with ZFC. In order to see this, let :

$$\mathbb{P} = (\{\text{finite partial functions } f : \omega \to 2\}, \leq)$$

We define $p \leq q$ if and only if p extends q. Since $\mathbb{P}$ is countable, it certainly has the c.c.c. We note that p is compatible with q if and only if p and q agree on the intersection of their domains. For each n, let $A_n = \{p \in \mathbb{P} : n \in \text{dom(f)}\}$ and for each $f \in 2^\omega$, let $B_f = \{p \in \mathbb{P} : \exists n \in \text{dom(p)} : p(n) \neq f(n)\}$.
We set $\mathcal{D} = \{A_n\} \cup \{B_f\}$ and note that $|\mathcal{D}| = 2^\omega$. Moreover, each A_n and B_f are easily seen to be dense in $\mathbb{P}$.

[13] Actually, Cantor proved in 1872 that any closed set of finite Cantor-Bendixson rank is a set of uniqueness. Lebesgue further extended the result for any countable closed set in 1903 and finally, Bernstein (1908) and Young (1909) extended the result for any countable set.

Now let G be a filter on $\mathbb{P}$ such that, towards a contradiction, $G \cap d \neq \emptyset$ for each $d \in \mathcal{D}$. We note that since G is a filter, given any $\Gamma \subseteq G$ there is an element $g_\Gamma \in \mathbb{P}$ which extends all elements in Γ. Hence, if we choose $\Gamma = \{G \cap d\}_{d \in \mathcal{D}}$ we can consider its extension $g_\Gamma \in \mathbb{P}$. But then dom(g_Γ)=ω even though it does not coincide with any $f \in 2^\omega$, which is a contraction. We can therefore conclude that $MA(2^\omega)$ is false in ZFC.

While proving that $MA(\omega)$ holds in ZFC (Theorem 4.89), we noted that the c.c.c. was not necessarily. However, if $\mathbb{P}$ is not c.c.c., then $MA(\omega_1)$ may fail in ZFC. Indeed, consider the following :

$$\mathbb{P} = (\{\text{finite partial functions } f : \omega \to \omega_1\}, \leq)$$

Again, $p \leq q$ if and only if p extends q. We note that $\mathbb{P}$ is not c.c.c. since $\{\langle 0, \alpha \rangle\}_{\alpha \in \omega_1}$ is a family of pairwise incompatible elements of $\mathbb{P}$. Moreover, for each $\alpha < \omega_1$ let $D_\alpha = \{p \in \mathbb{P} : \alpha \in \text{ran(p)}\}$ so that $\mathcal{D} = \{D_\alpha\}$ is a family of dense subsets of $\mathbb{P}$. Suppose that, towards a contradiction, G is a filter on $\mathbb{P}$ such that $G \cap d \neq \emptyset$ for each $d \in \mathcal{D}$. Setting $\Gamma = \{G \cap d\}_{d \in \mathcal{D}}$ and considering the extension $g_\Gamma \in \mathbb{P}$, we would have that $\text{ran}(g_\Gamma) = \omega_1$, which is clearly impossible.

The following result is a consequence of $MA(\kappa)$ with a strong combinatorial flavour. Despite its *technical* nature, perhaps even a bit *cumbersome*, it has remarkable *direct* applications. Here, we will use it to prove Proposition 4.92 and Corollary 4.91.

Proposition 4.90. Assume $MA(\kappa)$. Let $A, C \subseteq \mathcal{P}(\omega)$ be such that $|A|, |C| \leq \kappa$. Moreover, assume that for all $y \in C$ and all finite $F \subseteq A$, $|y \setminus \bigcup F| = \omega$. Then, there is some $d \subseteq \omega$ such that for all $x \in A$ one has that $|d \cap x| < \omega$ and for all $y \in C$, one has that $|d \cap y| = \omega$.

Proof. The reader can check a proof in [6] (Theorem 2.15, p.57). ∎

We take a brief detour in order to prove a result which can provide some insight on Martin's Axiom. In a sense, under $MA(\kappa)$ any cardinal $\omega \leq \kappa < 2^\omega$ behaves not too *wildly*.

Recall that $A \subseteq \mathcal{P}(\kappa)$ is said to be an almost disjoint family if each $x \in A$ is such that $|x| = \kappa$ and if for any two distinct elements $x, y \in A$ one has that $|x \cap y| < \kappa$.

Corollary 4.91. Assume $MA(\kappa)$. Then, $2^\kappa = 2^\omega$.

Proof. We start by stating the following claim :

Claim : Let $B \subseteq \mathcal{P}(\omega)$ be an almost disjoint family of size κ and let $A \subseteq B$. Then, there is some $d \subseteq \omega$ such that for any $x \in A$ one has that $|d \cap x| < \omega$ and for any $y \in B \setminus A$, one has that $|d \cap x| = \omega$.

We note that this Claim follows from Proposition 4.90, using $C = B \setminus A$.

We now fix any almost disjoint family B of size κ.[14] It is enough to define a surjective map $\Phi : \mathcal{P}(\omega) \to \mathcal{P}(B)$. We set the following :

$$\Phi(d) = \{x \in B : |d \cap x| < \omega\}$$

Finally, note that our Claim implies that Φ is onto. ■

In ZFC, the countable union of meager sets is meager and the countable union of null sets is null. We prove that under $MA(\kappa)$, we can extend this closure property for any $\omega \leq \kappa < 2^\omega$. Once again, this strenghtens the idea that under $MA(\kappa)$ every cardinal κ between ω and 2^ω, behaves *roughly* like ω.

Proposition 4.92. Assume $MA(\kappa)$. Then, the union of $\leq \kappa$ meager sets in a Polish space is meager.

Proof. It is enough to prove that whenever one has κ dense open sets, say $\mathcal{U}_\alpha$, one can always find countably many dense open sets, say $\mathcal{V}_n$, such that $\bigcap_{n\in\omega} \mathcal{V}_n \subseteq \bigcap_{\alpha\in\kappa} \mathcal{U}_\alpha$. Let $\mathcal{B} = \{B_n\}$ be a countable basis.
For each j, let $C_i = \{j \in \omega : B_i \subseteq B_j\}$ and take $C = \{C_j\}_{j\in\omega}$. On the other hand, for each $\alpha \in \kappa$ let $A_\alpha = \{i \in \omega : B_i \subsetneq \mathcal{U}_\alpha\}$ and take $A = \{A_\alpha\}_{\alpha\in\kappa}$.
By Proposition 4.90, let $d \subseteq \omega$ be such that $|d \cap C_j| = \omega$ and $|d \cap A_\alpha| < \omega$, for each $j \in \omega$ and $\alpha \in \kappa$. Now we define :

$$\mathcal{V}_n = \bigcup\{B_i : i \in d \text{ and } i > n\}$$

Each $\mathcal{V}_n$ is a non-empty open set. Moreover, it is dense : consider any $B_j \in \mathcal{B}$. Pick some $i \in d \cap c_j$ such that $i > n$, which is possible since $|d \cap c_j| = \omega$. It follows that $B_j \cap \mathcal{V}_n \neq \emptyset$. It remains to check that $\bigcap_{n\in\omega} \mathcal{V}_n \subseteq \bigcap_{\alpha\in\kappa} \mathcal{U}_\alpha$:
Note that for any α, $|d \cap A_\alpha| < \omega$. Hence, for any α there is some n such that for any $i > n$ one has that if $i \in d$, then $B_i \subseteq \mathcal{U}_\alpha$. Thus, for any α there is some n such that $\mathcal{V}_n \subseteq \mathcal{U}_\alpha$ and we are done. ■

Remark 4.93. Assume $MA(\kappa)$ and suppose that $X \neq \emptyset$ is a Polish space and $X = \bigcup_{\alpha<\kappa} F_\alpha$, with each $F_\alpha \subseteq X$. If each F_α is meager, then by Proposition 4.92 one has that X is meager. But this contradicts the Baire Category Theorem. In particular, X is not a union of $\alpha \leq \kappa$ nowhere dense closed sets.

Proposition 4.94. Assume $MA(\kappa)$ and let for each $\alpha < \kappa$, M_α be null subsets of $\mathbb{R}$. Then, $\bigcup_\alpha M_\alpha$ is null.

Proof. Fix any $\epsilon > 0$. Our goal is to find an open set $\mathcal{U}$ such that $\bigcup_{\alpha\in\kappa} M_\alpha \subseteq \mathcal{U}$ and $\mu(\mathcal{U}) \leq \epsilon$. We fix a countable basis $\mathcal{B} = \{B_n\}$ for $\mathbb{R}$. Furthermore, we define the following :

[14] Such family exists (c.f [6], Theorem 1.2, p.48)

$$\mathbb{P} = (\{p \subseteq \mathbb{R} : \ p \text{ is open and } \mu(p) < \epsilon\}, \leq)$$

Here, $p \leq q$ if and only if $q \subseteq p$. We note that p and q are compatible if and only if $\mu(p \cup q) < \epsilon$. We make the following observations :

(i) $(\mathbb{P}, \leq)$ is c.c.c. This can be seen using the separability of the measure space of $\mathbb{R}$. For a detailed proof, c.f [6] (Theorem 2.21, p. 59).

(ii) Suppose that G is a filter on $\mathbb{P}$. Then $\bigcup G \in \mathbb{P}$. Indeed, it is clearly an open set. Moreover, let $A = G \cap \mathcal{B}$. Clearly, $\bigcup A \subseteq \bigcup G$. Conversely, let $x \in p$ for some $p \in G$. Choose some $q \in \mathcal{B}$ such that $x \in q$ and $q \geq p$. It follows that $q \in G$, hence $q \in B \cap G$. Hence, $\bigcup A = \bigcup G$ and by countable additivity of μ (and using the compatibility condition and that G is a filter), it follows that $\bigcup A \in \mathbb{P}$.

(iii) Now for each $\alpha \in \kappa$, let $D_\alpha = \{p \in \mathbb{P} : M_\alpha \subseteq p\}$. Then, $\mathcal{D} = \{D_\alpha\}_{\alpha \in \kappa}$ is a family of dense sets. Indeed, let $q \in \mathbb{P}$ so that in particular, $\mu(q) < \epsilon$. Since each M_α is null, there is an open set $\mathcal{V}$ such that $M_\alpha \subseteq \mathcal{V}$ and $\mu(\mathcal{V}) < \epsilon - \mu(q)$. Hence, we take $p = \mathcal{V} \cup q$ and note that $p \in D_\alpha$ and $p \leq q$.

(iv) Finally, by $MA(\kappa)$ there is a filter G on $\mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$. Thus, $M_\alpha \subseteq \bigcup G$ and by (ii), we are done. ■

4.5.2 Proof of a generalization of Bari Theorem

What follows is a small modification on the proof of Theorem 20.1, in [18]. We need the following result on trigonometric series with bounded partial sums :

Proposition 4.95. (Vallee-Poussin) Let $S \sim c_n e^{inx}$ be a trigonometric series such that for each x, $S(x)$ has bounded partial sums. Then, if $\sum c_n e^{inx} = 0$ almost everywhere, it follows that $c_n = 0$ for all $n \in \mathbb{Z}$.

Proof. The reader can check a proof in [18] (Theorem 20.2). ■

Proof. (of Theorem 4.86) : Let $\{E_\alpha\}_{\alpha<\kappa}$ be a family of closed sets of uniqueness and let $E = \bigcup_\alpha E_\alpha$. Suppose that $\sum c_n e^{inx} = 0$ off E. We show that $c_n = 0$ for all $n \in \mathbb{Z}$.
By Corollary 3.24, $\mu(E_\alpha) = 0$ and by Proposition 4.94, $\mu(E) = 0$. It follows that, by Proposition 3.14, $c_n \to 0$.
Assume, towards a contradiction, that c_n is not identically zero. Define the following set :

$$G = \{x \in \mathbb{T} : \{S_N(x)\} \text{ is unbounded}\}, \text{ with } S_N(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

By Proposition 4.95, $G \neq \emptyset$. Note that $G \subseteq E$ and that G is a G_δ set of $\mathbb{T}$, hence G is a non-empty Polish space. Let $G_\alpha = E_\alpha \cap G$, which are closed sets in G. Since $G = \bigcup_\alpha G_\alpha$, it follows from Proposition 4.92 (and Remark 4.93) that there is some α_0 and some interval I_0 such that $I_0 \cap G = I_0 \cap G_{\alpha_0} \neq \emptyset$.

We prove that $\sum c_n e^{inx} = 0$ on I_0, thus $I_0 \cap G = \emptyset$, yielding a contradiction.

Choose some $f \in C(\mathbb{T})$ which is infinitely differentiable with $f > 0$ on I_0 and $f = 0$ off I_0.[15] Let $T \sim S(f).S$ so that $T \sim C_n e^{inx}$. Since f is infinitely differentiable and since $c_n \to 0$, it follows from Theorem 3.22 that :

$$\sum (C_n - f(x)c_n)e^{inx} = 0, \text{ for all } x$$

Thus, it is enough to show that $C_n = 0$ for each $n \in \mathbb{Z}$. Since E_{α_0} is a set of uniqueness, it is enough to show that $\sum C_n e^{inx} = 0$ for $x \notin E_{\alpha_0}$. So, let $x \notin E_{\alpha_0}$. Note that since $\sum C_n e^{inx} = 0$ off $I_0 \cap E$, one can assume that $x \in I_0 \cap E$. By regularity, there is some interval $J_0 \ni x$ such that $J_0 \subseteq I_0$ and that $\overline{J_0} \cap E_{\alpha_0} = \emptyset$.

Let $g \in C(\mathbb{T})$ be some infinitely differentiable function such that $g(x) = 1$ and $\mathrm{supp}(g) \subseteq \overline{J}$. We consider $R \sim S(g).T$, with $R \sim \sum D_n e^{inx}$. Note that $\sum c_n e^{inx} = 0$ almost everywhere and thus, $\sum C_n e^{inx} = 0$ a.e. which implies that $\sum D_n e^{inx} = 0$ a.e. On the other hand, $\sum D_n e^{inx}$ has bounded partial sums outside $\overline{J_0} \cap G = \overline{J} \cap G_{\alpha_0} = \emptyset$, since $\sum c_n e^{inx}$ and thus $\sum C_n e^{inx}$ has the same property. Hence, by Proposition 4.95 it follows that $D_n = 0$ for all n and consequently, by Theorem 3.22, we conclude that $\sum C_n e^{inx} = 0$. ■

Corollary 4.96. In ZFC, the union of countably many closed sets of uniqueness is a set of uniqueness.

Proof. This follows from Theorem 4.86 and Theorem 4.89. ■

Proof. (of Corollary 4.87) : Since $|E| \leq \kappa$, then E is a union of $\leq \kappa$ points. Each singleton is a closed set of uniqueness by Proposition 3.11, hence the result follows immediately from Theorem 4.86. ■

References

[1] A. Kechris, *Classical descriptive set theory*, GTM Springer-Verlag, 1995

[2] S. M. Srivastava, *A course on Borel sets*, GTM Springer-Verlag, 1998

[3] Y. Moschovakis, *Descriptive set theory : Second Edition*, AMS Mathematical Surveys and Monographs vol. 155, 2009

[4] J. C. Oxtoby, *Measure and Category : a survey of the analogies between topological and measure spaces*, GTM Springer-Verlag, 1980

[5] D. Martin, *A purely inductive proof of Borel determinacy*, Proc. Symp. Pure Math. 42, 1985 (303-308)

[15] For instance, using a *bump function*.

[6] K. Kunen, *Set theory : an introduction to independence proofs*, North Holland Publishing Co., 1980

[7] T. Jech, *Set theory - The Third Millennium Edition*, Springer, 2003

[8] S. Hachtman, J. Palumbo, *Notes on determinacy*

[9] J. Mycielski, S. Swierczkowski, *On the Lebesgue measurability and the Axiom of determinateness*, Fund. Math. 54, 1964 (67-71)

[10] L. Vuilleumier, *Continuous reductions on Polish spaces and quasi-Polish spaces*, Master thesis (sup. J. Duparc), E.P.F. Lausanne, 2016

[11] D. Martin, J.R. Steel, *A proof of projective determinacy*, Journal of Amer. Math. Soc., vol 2, 1989

[12] A. Kechris, A. Louveau, *Descriptive set theory and the structure of sets of uniqueness*, Cambridge University Press, 1987

[13] R. Kaufman, *Lipschitz spaces and Suslin sets*, Journal of Functional Analysis 43, 1981 (271-273)

[14] S. Glab, *On complexity of continuous functions differentiable on cocountable sets*, Real Anal. Exchange 34, 2008 (521-530)

[15] N. Sofronidis, *The set of continuous piecewise differentiable functions*, Real Anal. Exchange 31, 2005 (13-22)

[16] R. Mauldin, *The set of continuous nowhere differentiable functions*, Pacific Journal of Math. vol. 83, 1979 (199-205)

[17] R. Kaufman, *Fourier transforms and descriptive set theory*, Cambridge University Press vol 31, 1984 (336-339)

[18] A. Kechris, *Set theory and uniqueness for trigonometric series*

[19] P. Niemiec, *Borel parts of the spectrum of an operator and of the operator algebra of a separable Hilbert space*, Studia Math 208, 2012 (77-85)

[20] C. Badea, S. Grivaux, *Size of the peripheral point spectrum under power or resolvent growth conditions*, Journal of Functional Analysis 246, 2007 (302-329)

[21] B. Jamison, *Eigenvalues of modulus 1*, Proc. Amer. Math. Soc. vol 16, 1965 (375-377)

[22] P. Schlicht, *Continuous reducibility and dimension of metric spaces*, to appear in Archive for Mathematical Logic, 2017

[23] S. Mazurkiewicz, *Uber die menge der differenzierbaren functionen*, Fund. Math. 27, 1936 (244-249)

[24] A. Kechris, *Trigonometric series and set theory* (notes)

[25] R. van Wesep, *Wadge degrees and descriptive set theory, Wadge degrees and projective ordinals*, Cambridge University Press (Lecture notes in Logic) vol 37, 2012 (24-42)

[26] A. Andretta, *The SLO Principle and the Wadge Hierarchy*, College Publications, Fundations of the Formal Sciences V, Infinite Games, Studies in Logic vol.11, 2007 (1-38)

[27] D. Gale, F. M. Stewart, *Infinite games with perfect information*, Contributions to the theory of games, vol. 2. Annals of Mathematical Studies, vol.28, Princeton University Press (245–266), 1953

[28] D. Martin, *Borel determinacy*, Annals of Mathematics vol.102 (363-371), 1975

[29] H.G. Dales, W.H. Woodin, *An introduction to independence for analysts*, Cambridge University Press, 1987

[30] J.-B. Hiriart-Urruty, *Extension of Lipschitz functions*, J. Math. Anal. and Appl. 77 (539-554), 1980